ELEMENTS OF DYNAMICS

(KINETICS AND STATICS)

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WITH NUMEROUS EXERCISES

A TEXT-BOOK FOR JUNIOR STUDENTS

BY THE

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SIXTH IMPRESSION

LONGMANS, GREEN, AND CO.
39 PATERNOSTER ROW, LONDON
NEW YORK AND BOMBAY
1906

PREFACE

IN the following pages I have tried to exhibit the Elements of Dynamics in the simplest form which the subject seems to admit. It is throughout a strictly elementary work, and is designed to be put into the hands of any Student who has read the first Four and the Sixth Books of Euclid's Geometry, Algebra to the solution of Quadratic Equations, and Plane Trigonometry as far as the solution of Triangles. It seems to me desirable that a boy when equipped with this amount of Elementary Mathematics, should enter on the study of Dynamics somewhat earlier in school life than is now the custom.

In placing at the threshold the pure Geometry of Motion, irrespective of the cause of that motion, and then founding the entire science of Dynamics, in its Kinetic and Static branches, on Newton's 'Three Laws of Motion,' I have, like many recent writers on the subject, followed the system advocated by Professors Thomson and Tait; and there seems to be now an almost universal consensus of opinion that this is the most philosophical and the simplest method of proceeding.

The present Work may be considered as divided into two Parts—the First Part embracing Chapters i-xi., containing most of the principles, and the Second Part, included in Chapters xii-xvi., containing applications of those principles.

The following Articles seem to constitute a suitable course for a first reading of the book :—

Articles 1-30; 43-54; 61-75; 80-111; 124-130; 135-149; 156-178; 181-186; 188; 189; 194-208; 220-254.

For a second reading, the chapter on Impact may be taken with the Articles previously omitted in chapters i.-xi.; and then, as a final course, the chapters on Projectiles, Friction, Work, and Energy may be studied.

Thus treated, this book will serve the useful purpose of leading to more difficult and more exhaustive works, such as Garnett's Elementary Dynamics, and Minchin's Treatise on Statics.

It is not necessary that all the Examples in each set should be worked. Those at the end of some groups are more difficult than the earlier, and requiring, it may be, special skill in applying principles, may be omitted with advantage until the Student has made some progress in the subject.

The Examples have been chiefly taken from the Examination Papers set in the Royal Naval College during the past thirteen years. Some have been put together to illustrate particular points, some have been suggested by those in other works, and others may be regarded as common property, appearing in almost every collection. They are intended to form a special feature in this book; and besides their unusual number (in typical questions worked as models, and in problems proposed for solution), they have been carefully arranged with the view (1) of presenting several examples of the same kind in succession as in Text-Books of Arithmetic, Algebra, and Plane Trigonometry, (2) of leading from the easier to the more difficult, and (3) of bringing out the Student's grasp of principles.

Throughout both Text and Exercises a distinction has been observed between the expressions so many *pounds weight*, and so many *pounds*—the former always denoting a certain *force*, the latter invariably meaning a certain *quantity of matter*.

While designed to cover the course required in the Examination for the rank of Lieutenant in the Royal Navy, the work includes the extra subjects expected from candidates for the Beaufort Testimonial; and it is my hope that it may be found useful by those reading for the Oxford and Cambridge Local Examinations, as well as by those who are studying the subject in the ordinary course of education.

There remains the pleasant duty of acknowledging the generous help given me in its preparation.

A friend directed my attention to the beautiful proof of Varignon's Theorem of Moments given in Article 186, and which, so far as I know, has not appeared before in any Elementary Treatise. I am under very great obligations to Mr. Fletcher, of the Royal Naval College, for much assistance and advice in most parts of the book. He has moulded by his many suggestions the form of the first four Chapters, and especially my treatment of the Second Law of Motion. To Mr. Goodwin, R.N., also of the Royal Naval College, I owe much. His long experience as an Examiner enabled him to suggest many improvements in those parts where young students are found to fail most, and many of the Articles have been re-written by his advice. All the sheets have passed under his eye. I wish to say here that, without the help of these friends, this book would not be such as it now is, and for their great kindness I heartily thank them.

The work has been a difficult one to see through the press, and I am not so sanguine as to hope that all errors have been corrected. I shall feel indebted for any lists of such which may be sent me, and suggestions, especially from Teachers, for the improvement of the book will be most gratefully acknowledged.

JOHN LOVELL ROBINSON.

ROYAL NAVAL COLLEGE,
25th February 1888.

PREFACE TO THE SECOND EDITION

I HAVE carefully revised the Text and the wording of all the Examples. The retention in a few cases of the expression 'weighing so many pounds' in preference to the unusual one 'massing so many pounds' will I hope not lead the young Student to confuse the distinct ideas of 'Weight' and 'Mass.'

A few alternative proofs have been inserted, and others have been simplified. I advise young Students *to avoid Formulæ when possible.* In this subject, very few indeed need be committed to memory. Thus examples in Pulleys are easily worked from figures by first principles. The addition of interesting and easy Examples in Relative Velocity, Change of Units, Impact, Pile-driving, Trains in Motion, Horse-power, Work, Energy, and Motion in a Circle, will probably be appreciated by both Instructors and Students.

The book owes a good deal to the many valuable suggestions which have reached me. I very thankfully acknowledge the interest thus shown, and I shall be glad of like kindly help in improving the present edition.

J. L. R.

ROYAL NAVAL COLLEGE,
16*th May* 1890.

PREFACE TO THE FIFTH EDITION

THIS work has now assumed its final form. In successive editions I have inserted Appendices containing Duchayla's Proof of the Parallelogram of Forces, and two different proofs that the Path of a Projectile is a Parabola; some important typical examples have been fully worked out, and more than three hundred Exercises have been added from recent Examination Papers set in the Royal Naval College at Greenwich. The Examples have always formed a special feature of this book, and in selecting them I have taken great care that they should be representative, interesting, and really *workable* by students of average ability. I have excluded those which are manifestly fitted only for the use of advanced men.

J. L. R.

ROYAL NAVAL COLLEGE,
19*th October* 1901

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INTRODUCTION

OUR knowledge of the laws of nature has become accurate, and can be extended by careful observation and experiments alone. In such investigations the conceptions of Time, Space, Mass, Force, and other elements are required. We assume that certain of these are known intuitively, and from them we deduce others.

Those usually taken as fundamental are Time, Space, and Mass; and the next step is to devise instruments for their accurate measurement. Time is measured by clocks, chronometers, chronoscopes, etc.; Space is measured in its linear dimensions by rods, chains, etc., and in its angular dimensions by sextants, theodolites, etc.; and Masses are compared by balances. But no instrument can give us practical measures of anything without reference to certain Standards or Units of that thing by which they can be estimated and compared.

The Standard of time furnished by nature in the first instance is the *Sidereal Day*, which is defined to be the period of the earth's rotation on its axis. From this is derived the *Mean Solar Day*, which is the mean of the intervals which elapse between several successive transits of the sun across the same meridian. For practical purposes this may be regarded as constant. A good chronometer will mark 24 hours in the Mean Solar Day. Each hour contains 60 minutes, and each minute contains 60 seconds.

Therefore the *Mean Solar Second* is the $\frac{1}{86400}$ part of a Mean Solar Day, and this is the **Unit of Time** made use of in physical research in all civilised countries.

The **Unit of Length** used in England is the *Imperial Yard*. This length was not originally derived from any data supplied by nature,[1] although some important relations with such have been discovered, and by which it could be reproduced if necessary.

By English law the Yard is 'the distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Office of the Exchequer,' when the whole is at a temperature of 60° Fahr. If this Standard Yard should be lost or destroyed, it may be restored from the authorised copies of it which are kept in the Mint, in the Houses of Parliament, in the Royal Observatory, and in the custody of the Royal Society.

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[1] There is a tradition that this measure was derived from the length of Henry I.'s arm.

A *Foot* is the one-third part of the Imperial Yard.

The Unit of length in France is the *Mètre*. This was originally supposed to be the ten-millionth part of the length of an arc of the earth's meridian between the Pole and the Equator, and was made the National Standard by a decree of the French Republic in 1795. It is legally defined as 'the distance, at the temperature of melting ice, between the ends of a platinum rod preserved in the Archives.' It is practically defined by certain accurate Mètres which are kept in various places, and by comparison with which it could be restored in case of accident. The original Standard Mètre was made by Borda.

1 Mètre = 39·37043 inches.

1 Centimètre = 0·3937043 inches, or nearly $\frac{2}{5}$ inch.

1 Kilomètre = 39370·43 inches, or nearly $\frac{5}{8}$ mile.

The **Unit of Mass**[1] in England is the quantity of matter in an Imperial Standard Pound Avoirdupois (*see* Art. 87). The Standard Pound is a piece of platinum preserved in the Exchequer Office, and marked 'P. S. 1844. 1 lb.' If lost or destroyed, it may be restored from authorised copies of it preserved with the authorised copies of the Imperial Yard.

1 Pound = 7000 grains.

The French Unit of Mass is the 'Kilogramme des Archives.' This was originally intended to be the mass of a litre of distilled water at a temperature of 4° C., but is always practically defined by authorised copies of the standard.

1 Cubic foot = 28·3 litres.
1 Kilogramme = 1000 grammes.
1 Kilogramme = 2·204 pounds.
1 Gramme = 15·43 grains.

In physical research the units of Space, Mass, and Time, usually employed are the centimètre, gramme, and second, and this system of units is known as the 'C. G. S.' system.

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[1] 'Mass' is a technical word used to denote the *quantity of matter* in a body (*see* Art. 83).

There are many aspects of Motion which may be considered quite apart from the physical ideas involved in force, mass, elasticity, etc. Ampère has suggested the term *Kinematics*[1] to denote the purely geometrical *Science of Motion;* and in all recent treatises dealing with 'Matter and Motion,' whether elementary or advanced, this Geometry of the subject is made introductory to **Dynamics** or the *Science of Force.* The latter will naturally be regarded in two aspects, as the forces under consideration cause motion or maintain rest.

The former branch of Dynamics is called *Kinetics,* the latter branch is called *Statics.* These terms may be formally defined as follows:—

Kinematics is the Science of Motion without reference to the masses of the bodies moved, or to the causes which produced the motion.

Kinetics is that division of Dynamics which treats of the action of force producing motion.

Statics is that division of Dynamics which treats of the action of force maintaining rest.

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[1] Κίνημα = motion.

It may be here remarked that the subject-matter of the present work has often been presented under the title **Mechanics**. Now this word properly denotes the Science of Machines; in this sense it was used by Newton. There is, at present, a marked tendency to revert to the original meaning of the word.

CHAPTER I.

UNIFORM VELOCITY.

1. THE *Motion* of a point is its 'change of position.'

The **Velocity** of a point is its 'rate of motion,' or its 'rate of change of position.'

To say how fast a point is going is to assert something about its state of motion at *a particular instant.* The student must at the very beginning realise that the velocity of a moving point is an *instantaneous* property of the motion which may or may not change from instant to instant.

2. **Uniform Velocity** is the rate of motion of a point which remains *constant from instant to instant*, and as a consequence, a point moving with uniform velocity will traverse equal spaces in equal times, however small those times may be, or, in other words, is always moving at the same rate.

It is not sufficient to define uniform velocity as 'a velocity with which equal spaces are traversed in equal times,' because a train, for example, may describe 20 miles an hour, and yet at different periods of each hour its velocity may not be the same.

3. **Variable Velocity** is the rate of motion of a point which is *not the same from instant to instant.*

4. We have seen that a velocity is a property of motion which belongs to it at any given instant, but it is evident that we cannot *measure* it instantaneously. To find how fast a person is walking at any moment, he must keep on

walking *at that rate* for some definite time, and thus discover how far he can go in a certain time.

We see then that a velocity involves the ideas of *space* and *time.*

5. The units of space and time used in England are a *Foot* and a *Second.* This system of units is known as the 'foot-second,' or simply as the 'f.-s.' system.

In France a Centimetre and a Second are the units used.

6. The **Unit Velocity** is 'the uniform velocity with which the unit space is described in unit time.'

The British Unit Velocity is therefore 'the uniform velocity with which a foot is described in a second,' or, briefly, 'a velocity of a foot per second.'

7. Now since any quantity is *measured* by the number of units of that quantity which it contains, we infer that **Uniform Velocity is measured** by the space passed over in a second.

Thus a uniform velocity of 10 f.-s. means that the point will describe 10 feet during every second of its motion.

8. Of course velocity may be measured by any units of space and time which may be considered suitable : but if we desire to *compare* the velocities with which two points are moving, we must express their measures in terms of the same units.

9. *If a point move with a uniform velocity* (v) *for a given time* (t), *to find the whole space described.*

By Def. :—

in 1 sec. the point describes v feet from its initial position.
„ 2 „ „ „ $2v$ „ „
„ 3 „ „ „ $3v$ „ „
„ t „ „ „ tv „ „
But the whole space described $=s$.

$$\therefore s=vt.$$

This is the *fundamental equation of uniform motion in a straight line.* In it three quantities are connected, and if any two of them be given the third may be found. It is evident, by Algebra, that if v be constant, the space described varies directly as the time of motion.

Example.—A point moves with a uniform velocity of 10 f.-s.; find the space described in a minute.

$$\text{Here } v = 10, \text{ and } t = 60,$$
$$\therefore\ s = 10 \times 60 = 600 \text{ feet.}$$

10. Since a quantity is measured by the number of units of that quantity which it contains, we see that

$$\text{Quantity} = \text{Measure} \times \text{Unit.}$$

Hence we infer, by Algebra, that the measure of any definite quantity will *vary inversely* as the unit employed.

11. Suppose that v is the measure of any velocity in the 'foot-second' units, and we require the measure of the same velocity when x *feet* are chosen as the unit of space, and t *seconds* as the unit of time, we may work from our definition as follows:—

Let v_1 denote the measure required.

By Def. v_1 means that the point will describe

$$v_1 \text{ times } x \text{ feet per } t \text{ secs.};$$
$$\text{or, } v_1x \text{ times 1 foot per } t \text{ secs.};$$
$$\text{or, } \frac{v_1x}{t} \text{ times 1 foot per 1 sec.}$$

But by Hyp. v means that the point will describe v times 1 foot per 1 sec.

$$\therefore\ \frac{v_1x}{t} = v;$$
$$\therefore\ v_1 = \frac{vt}{x}.$$

If v be constant, then by Algebra, $v_1 \propto \frac{t}{x}$, *i.e.* the measure of a velocity will vary directly as the unit of time employed, and inversely as the unit of space.

Example.—If the velocity of a point be 44 f.-s., find its measure in the mile-hour system of units.

Using the formula just established, we have

$$v = 44,\ t = 3600,\ s = 5280;$$

$$\therefore\quad = \frac{44 \times 3600}{5280} \text{ miles per hour;}$$

$$= 30 \text{ miles per hour;}$$

or thus, independently of the formula,

$$\begin{aligned} &\text{A velocity of 44 feet per second,}\\ \equiv\ &\text{,,}\quad 44 \times 3600 \text{ feet per hour,}\\ \equiv\ &\text{,,}\quad \frac{44 \times 3600}{5280} \text{ miles per hour,}\\ \equiv\ &\text{,,}\quad 30 \text{ miles per hour,} \end{aligned}$$

i.e. a velocity of 44 feet per second is the *same* as a velocity of 30 miles per hour.

EXAMPLES—I.

✓ 1. A point is travelling with a velocity of 88 f.-s.; find its measure in the mile-hour system.

2. A point has a velocity of 15 miles an hour; find its measure in the foot-second system.

✓ Find the measure of a velocity of 30 miles an hour in the foot-second system.

✓ 3. Compare a velocity of 12 f.-s. with a velocity of 10 yards per sec.

✓ 4. Find the measure in the yard-minute system of a velocity of 45 f.-s.

✓ 5. A velocity of x feet per minute is four times a velocity of 3 yards per second; find x.

6. A velocity of 20 miles an hour contains the velocity of 33 feet per half-second v times; find v.

7. A man walks with a velocity the measure of which is 5·5, and finds that he has travelled 8 miles in 2 hours; find the unit of time, the unit of space being 2 feet.

8. A man walks with a velocity represented by 2, and he walks 7 miles in 2 hours; if 1 foot be the unit of length, find the unit of time.

9. A man walks with a velocity the measure of which is 15, and he

walks 10 miles in $3\frac{3}{8}$ hours; find the unit of space used, if the unit of time be $7\frac{1}{2}$ seconds.

10. A point describes a feet in n seconds; if unit of time be 1 minute, find unit of space in order that the numerical value of the point's velocity may be 1.

11. If v be the measure of a velocity using l feet and t seconds as units, find its measure when l_1 feet and t_1 seconds are the units.

12. If 2 seconds be the unit of time, and an acre be represented by 40, what number will represent a velocity of 30 miles an hour, when a yard is the unit of space?

12. **Variable Velocity is measured** at any instant by the space which would be passed over in a second, if the velocity at the instant considered remained the same throughout the second.

If we say that a train is travelling at the rate of 40 miles an hour, we do not necessarily imply that it has travelled 40 miles during the previous hour, or that it will travel 40 miles during the next hour, but that at the instant referred to the velocity is such that *if it remained the same for a whole hour* the distance described would be 40 miles.

13. If a point has described a certain space with variable velocity in any given time, we call the uniform velocity with which the same space might be described in that time the '**mean velocity**' or the '**average velocity**' of the point.

The *mean velocity* may practically be found by dividing the whole space described by the time taken to describe it.

A *nautical mile* is a mile of Latitude, *i.e.* $\frac{1}{60}$ degree on the earth's surface. It is usually taken as equal to 6080 feet.

DEF.—A *knot* is a velocity of a nautical mile per hour.

EXAMPLES—II.

1. A point moves with a uniform velocity of 12 f.-s.; find the whole space described in half a minute.

2. A train moving uniformly describes 240 miles in 6 hours; find its velocity in feet per second.

3. Compare the velocities of 2 points, one of which describes 5 feet per ½-second, the other 100 yards per minute.

4. Compare the velocities of 2 points, one of which describes 720 feet per minute, and the other 3½ yards per ¾-second.

5. Two points starting simultaneously from a given position in the same direction move one with a velocity of 10 f.-s., the other with a velocity of 15 f.-s.; find their distance apart at the end of 5 minutes.

6. Sound travels at the rate of 1090 feet per second. The report of a gun fired at sea is heard on shore after 35 seconds; how far is the ship from the shore?

7. A steamer going 12 knots steams at this rate for 4 days, and then altering her speed travels for 3 days 7 hours at the rate of 13 knots; find the whole distance travelled.

8. A train takes 7 hours 31 minutes to travel 205 miles; find its mean velocity in the foot-second system.

9. A point moves at the rate of 754 yards per hour; find the velocity in feet per second.

10. A train starts on a journey of 347 miles and travels at the rate of 30¼ miles an hour; after travelling 261 miles, the train is stopped by an accident for 2 hours 27 minutes, and then proceeds at the rate of only 16 miles an hour; find time of travelling the whole distance.

11. A train performs a journey of 45 miles in 2 hours; find the mean velocity in feet per second.

12. A point is moving at the rate of 2700 yards per minute; express this velocity in feet per second.

13. How long will it take to circumnavigate the earth at the rate of 12 knots?

14. The motion of a point at any instant is known when its **direction** and **rate** are given. We may therefore completely represent a velocity by a straight line: first, *in direction*, because we can always draw a straight line in the direction of the motion; next, *in magnitude*, because we can always make the line contain as many units of length (according to any determined scale) as there are units of velocity.

15. We thus bring velocities within the grasp of Geometry. By using the conventions regulating the signs + and − to denote contrariety of direction, we are able to indicate velocities in the same or contrary directions.

If AB denotes a velocity v, then BA will denote a velocity $-v$.

If 12 f.-s. mean that a point is moving due N. (suppose) with a velocity of 12 feet per second, then −5 f.-s. will indicate that a point is travelling due S. with a velocity of 5 feet per second.

All lines which have the same length and direction will represent the same velocity.

COMPOSITION AND RESOLUTION OF VELOCITIES.

16. A person walking fore-and-aft in a steamer going through still water, or moving from one window to another of a railway carriage in motion, or going aloft as a ship sails obliquely through a current, will present a case of what are called simultaneous velocities. In such instances the velocities are called *Component* velocities, and the actual velocity in space of the body possessing them is known as the *Resultant* velocity.

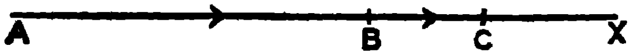
The **Composition of Velocities** is the determination of the Resultant when the components are given.

The **Resolution of Velocities** is the determination of Components when the Resultant is given.

17. *To find the Resultant of two velocities which are in the same straight line.*

Case I.—If they have the *same* direction, the Resultant is equal to their *sum.*

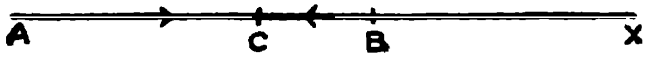
If a point move with a velocity v_1 in the direction of the straight line *AX* such that it will describe the distance *AB* in a second, and if the line *AX* moved in the same direction in the plane of the paper with a velocity v_2 sufficient to carry it the distance *BC* in a second, then it is evident that the point has moved from its initial position a distance *AC*, or (v_1+v_2) feet in a second. If v denote the resultant velocity, then



$$v=v_1+v_2. \quad (\text{Q.E.D.})$$

Case II.—If they have *opposite* directions the Resultant is equal to their *difference*.

If v_1 would carry the point from *A* to *B* in a second, and the line itself moving with velocity v_2 in the opposite direction were carried in the plane of the paper backwards through the distance *BC* in a second, then it is evident that the point has moved from its initial position a distance *AC* or (v_1-v_2) feet in a second.



If v denote the resultant velocity, then

$$v=v_1-v_2. \quad (\text{Q.E.D.})$$

COR.—If the components are *equal* and *opposite*, then $v=v_1-v_2=0$, and we say that the particle is at rest.

We mean at rest *relatively to the earth* or other selected object; such a thing as *absolute rest* is beyond our power to determine.

NOTE.—If a point have several velocities, some in one direction and the rest in the opposite direction, we can apply the above cases to determine their Resultant.

EXAMPLES—III.

1. Find the resultant of the following velocities in the same straight line:—

   (*a*) 5 and 7 in the same direction.
   (*b*) 12 to the east, and 3 to the east
   (*c*) 10 to the N. and 6 to the S.
   (*d*) 8, 10, 15 in one direction, 6, 7, 11 in the opposite direction.

2. Compound the velocities 3, 10, 13, -5 in the same straight line.

3. A sailor is running forward on the deck with a velocity of 10 feet per second; the ship is steaming 10 miles an hour in still water. Find the man's velocity over the ground.

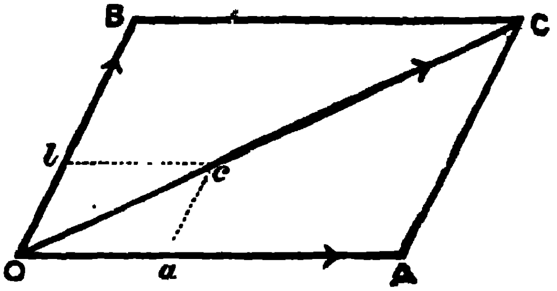
4. If in Ex. 3 the sailor is running aft, find his resultant velocity.

PARALLELOGRAM OF VELOCITIES.

18. The next case to be considered is that where a point possesses two velocities whose directions include a given angle. Their resultant may be found by the following important proposition, which is known as the '**Parallelogram of Velocities**':—

STATEMENT.—*If two component velocities be represented in magnitude and direction by two adjacent sides of a Parallelogram, their resultant velocity is represented in magnitude and direction by the diagonal which passes through their intersection.*

Let OA, OB, represent the components in magnitude and direction. Complete the parallelogram $OACB$. Then OC will represent the Resultant in magnitude and direction.



In a second the point moving along OA will arrive at A. During that second the line OA moving parallel to its initial position, with one extremity in the line OB, *and carrying the point in it*, will reach the position BC;

∴ at the end of the second the point will be at C.

We must next show that at any instant during the second the point is found in the line OC.

Assume any time τ less than a second.

Suppose that in the time τ the point moving with the given velocities would describe Oa and Ob respectively.

Complete the parallelogram *Oacb*, and join *Oc*. Then, as before, the point will be at *c* at the end of the time τ.

Now the velocities being uniform and the spaces described being therefore proportional to the times of describing them (Art. 9), we have

$$OA : Oa = OB : Ob = 1 : \tau.$$

$\therefore$ $OA : OB = Oa : Ob$, and the angle between the sides of the parallelograms being the same, we must have the diagonals OC and Oc coincident in direction, by Euc. vi. 26;

$\therefore$ the point is always found *in* the line OC;

$\therefore$ OC will represent the resultant in *direction*.

And since at the *end* of the second the point is at C;

$\therefore$ OC will represent the resultant in *magnitude*,

and this completes the proof.

Cor.—The parallelograms $OACB$, $Oacb$ being similar, it follows that $Oc : OC = Oa : OA = v\tau : v \cdot 1 = \tau : 1$, *i.e.* the distance described along OC being proportional to the time, we infer that the velocity of the point in the direction OC is uniform.

Note.—By the geometry of the parallelogram it follows that:—

1. The direction of the Resultant of two equal velocities bisects the angle between their directions.
2. The direction of the Resultant of two unequal velocities makes the less angle with the greater component.
3. If the diagonal AB intersect the diagonal OC in D, the Resultant will be completely represented by $2OD$.

19. *To express the Resultant* (v) *of two velocities* (v_1 *and* v_2) *in terms of the components and the included angle* (θ).

In Trigonometry it is shown that

$$OC^2 = OA^2 + AC^2 + 2OA.AC \cos\theta.$$

$$\text{or,} \quad v^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos\theta.$$

Cor. I.—If $\theta = 90°$, then $\cos\theta = 0$, and the formula becomes $$v^2 = v_1^2 + v_2^2,$$
as it ought by Euclid i. 47.

COR. II.—*As the angle between the directions of two components increases the Resultant diminishes.*

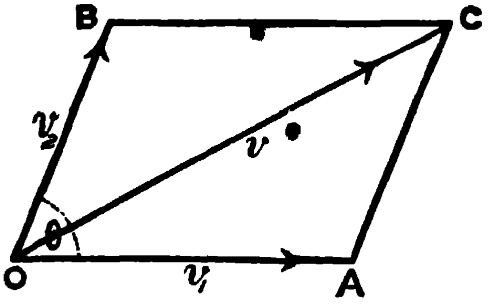


FIG. i.

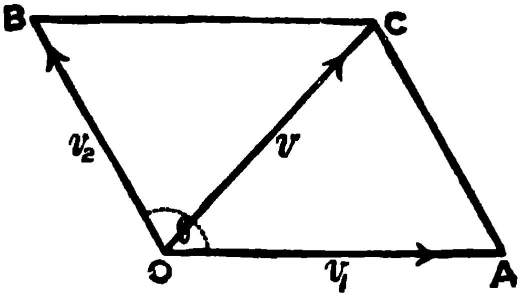


FIG. ii.

Let $\theta=0°$, or $90°$, or $180°$ successively; then $\cos\theta=1$, or 0, or -1 successively, and $v^2=v_1^2+v_2^2+2v_1v_2$; or $=v_1^2+v_2^2$; or $=v_1^2+v_2^2-2v_1v_2$ successively; which proves the theorem.

It may also be established by Geometry, thus:—

In the above figures, *OA* and *AC* of the triangle *OAC* are equal each to each, and the angle *OAC* in Fig. i. is greater than the angle *OAC* in Fig. ii.

∴ by Euc. i. 24, *OC* in Fig. i. is greater than *OC* in Fig. ii. (Q.E.D.)

EXAMPLES—IV.

1. A point tends to move with two equal velocities of 10 f.-s. in directions inclined at an angle of 120° to each other; find its path and the resultant velocity.

2. A point is subject to two velocities of 30 f.-s. and 40 f.-s. at right angles to each other; find the magnitude of the resultant.

3. An engine is running through a tunnel at the rate of 30 miles an hour, and the boiler suddenly bursts. A piece of metal is thrown off at right angles to the direction of the previous motion with a velocity of 33 f.-s. With what horizontal velocity will it strike the side of the tunnel?

4. A ball is rolled across the deck of a ship sailing 4 miles an hour with a velocity of 10 f.-s.; find its path and resultant velocity.

5. Two velocities of 12 f.-s. and 16 f.-s. are inclined at an angle of 60°; find the magnitude of their resultant.

6. Two velocities of 18 f.-s. and 21 f.-s. include an angle of 135° between their directions; find the magnitude of their resultant.

7. Two velocities, 12 and 15, are impressed on a point; find the greatest and least possible resultant.

8. The resultant of two equal velocities impressed on a point is 11 f.-s., and the angle between the components is 60°; find the value of the components.

9. If velocities of 2 and 3 have a resultant $=\sqrt{19}$, find θ.

10. If two velocities, 53 and $65\sqrt{2}$, include an angle of 45°, find the magnitude of their resultant.

11. If the resultant $=55$, the sum of the components $=60$, and $\theta=60^\circ$, find the components.

12. If $\theta=150^\circ$, find the ratio $v_1 : v_2$ if v is equal to the smaller component.

13. If $v_1=75$, $v_2=35$, and $\theta=\tan^{-1}(\frac{3}{4})$, find v.

14. Two velocities, 110 and 86·6, have a resultant of 190; find the angle between the components.

15. If $v_1=x(\sqrt{5}+1)$, $v_2=x(\sqrt{5}+1)$, $v=2x$, find θ.

16. If v_1 and v_2 are inclined at an angle of 120°, and v is at right angles to v_1, find v and v_2 in terms of v_1.

17. Two given velocities include an angle $=\theta$; if the angle $=\frac{\theta}{2}$, then resultant $=7v$; if $\theta=60^\circ$, resultant $=5v$; find θ.

18. If two velocities are in the ratio 3 : 4, and their resultant is a mean proportional between them, find θ.

19. Two velocities include an angle θ and their resultant $=v$; if the angle $=\frac{\pi}{3}-\theta$, the resultant $=\frac{v}{2}$; if the angle $=\frac{\pi}{3}+\theta$, the resultant $=\frac{v}{3}$; find θ.

20. The resultant of two equal velocities $=v$ when $\theta=60^\circ$; find the cosine of the angle between them when resultant $=\frac{v}{2}$.

21. If $v_1=a^2-b^2$; $v_2=2ab$; $v=a^2+b^2$; find θ.

22. Find the ratio $v_1 : v_2$, if v_1 be greater than v_2, the resultant $=v_2$, and $\theta=135^\circ$.

23. The resultant of two velocities at right angles is represented by 20, and it makes an angle of 30° with one of them; find the two components.

24. Velocities of 40 and 24 have a resultant $=v$, and the resultant is inclined at 90° to the 24; find v.

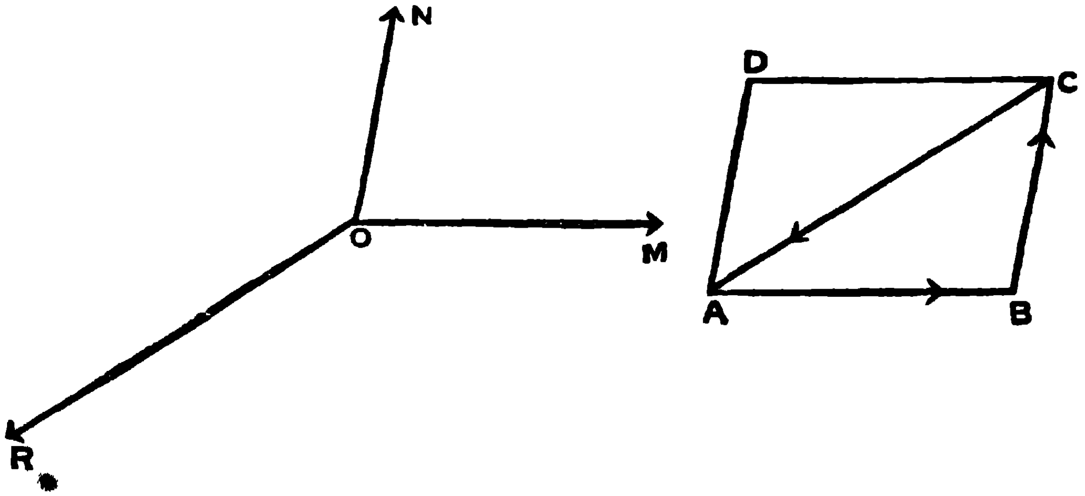
25. If two velocities impressed on a point at O be represented in magnitude and direction by OA and nOB, show that their resultant is represented by $(n+1)OC$, when C is a point in AB such that $AC=nBC$.

26. Two points starting instantaneously from the same initial position move uniformly with the same velocity along lines represented in direction by two sides of an equilateral triangle; find their distance apart after a given time.

27. Two points move with velocities v_1 and v_2 along lines inclined at an angle θ; find their distance apart after t seconds.

THE TRIANGLE OF VELOCITIES.

20. STATEMENT.—*If a point have three Component Velocities which are represented in magnitude and direction by the sides of a triangle taken in order, the point is at rest.*



If a point O have three velocities, OM, ON, OR, represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC, the point is at rest.

Complete the parallelogram $ABCD$. Then AD will also completely represent the velocity ON.

The resultant of OM and ON, *i.e.* of AB and $AD \equiv AC$ (Art. 18).

$\therefore$ the resultant of OM, ON, $OR \equiv$ the resultant of AC and CA, and this $\equiv$ Zero (Art. 17, Cor.).

∴ the resultant of the velocities represented by AB, BC, CA being zero, the point O will be at rest. (Q.E.D.)

NOTE 1.—It is hoped that this method of proving the proposition will prevent the student from falling into the error of supposing that the point subject to the velocities represented by the sides of the triangle ABC tends to move along the sides AB, BC, CA of the triangle.

NOTE 2.—The resultant of two velocities represented by AB and BC will be represented by AC.

21. The **converse of the Triangle of Velocities** is true, viz.: 'If a point be at rest when possessed of three velocities, we may represent those velocities in magnitude by the sides of *any* triangle drawn parallel to their directions.'

This follows by Euc. vi. 4.

EXAMPLES—V.

1. Show that velocities represented by 5, 6, 13, cannot keep the point at rest.

2. Can a point be at rest when possessed of velocities 7, 9, 12?

3. Find the resultant of velocities 4, 4, 4 acting parallel to the sides of an equilateral triangle taken in order.

4. Find the resultant of velocities 4, 5, 6 acting parallel to the sides of an equilateral triangle in order.

5. Find the resultant of velocities 8, 11, 20 which include angles of 120° with each other in order.

6. A point which possesses three velocities 3, 4, 5 is at rest; find the angle between 3 and 4.

7. A point is at rest when possessed of velocities 2, 3, $\sqrt{6}$; find the angle between 2 and 3.

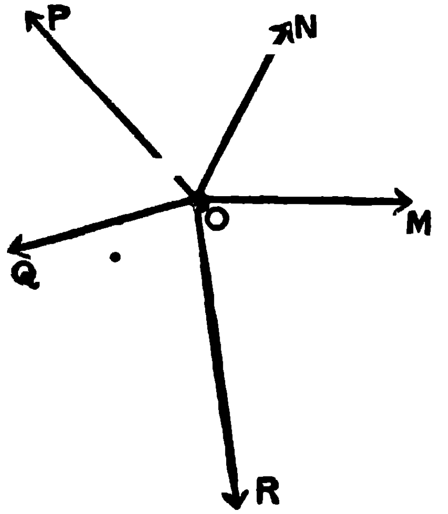
8. Show that a point is at rest when possessed of 3 velocities whose directions are perpendicular to the sides of a triangle, and whose magnitudes are represented respectively by the sides to which the velocities are perpendicular.

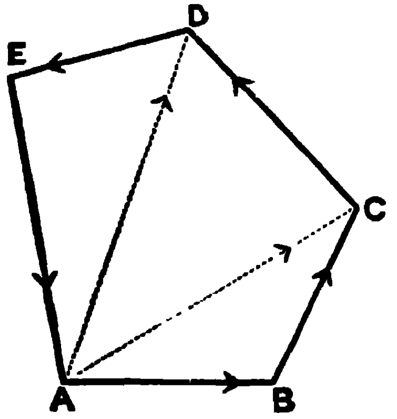
9. If three velocities v_1, v_2, v_3, keep a point at rest, and we have given the magnitude and direction of v_1, the magnitude only of v_3, and

the direction only of v_2, find the direction of v_3 and the magnitude of v_2.

THE POLYGON OF VELOCITIES.

22. STATEMENT.—*If a point have several Component Velocities which are represented in magnitude and direction by the sides of any polygon taken in order, the point is at rest.*





Let the velocities OM, ON . . . be represented in magnitude and direction by AB, BC, CD, DE, EA; then the point is at rest.

Draw AC and AD.

The resultant of OM, $ON \equiv$ resultant of AB, $BC \equiv AC$. (Art. 20, Note 2.)

$\therefore$ The resultant of OM, ON, $OP \equiv$ resultant of AC, CD $\equiv AD$.

$\therefore$ The resultant of OM, ON, OP, $OQ \equiv$ resultant of AD, DE $\equiv AE$.

$\therefore$ The resultant of OM, ON, OP, OQ, $OR \equiv$ resultant of AE, $EA \equiv$ Zero. (Art. 17, Cor.)

Therefore the point O is at rest. (Q.E.D.)

COR.—If a point have several component velocities represented in magnitude and direction by *all but one* of the sides of a polygon taken in order, their resultant is represented in magnitude and direction by that remaining side taken in the reverse direction.

If the velocities *OM*, *ON*, *OP*, *OQ* be represented in magnitude and direction by *AB*, *BC*, *CD*, *DE*, then it may be shown by the method given above that their resultant is represented in magnitude and direction by *AE*, which proves the proposition.

23. If then, without removing the pen from the paper, we draw straight lines equal and parallel to the several component velocities which a point may possess, we shall find that our lines will either form a *closed figure*, or they will not: if they do form a closed figure, then we infer that the Resultant is zero, and therefore that the point is at rest: if they do not form a closed figure, then we infer from the Corollary just stated that the line *necessary to close the figure, and drawn in the opposite way to our other lines*, will represent the Resultant in magnitude and direction.

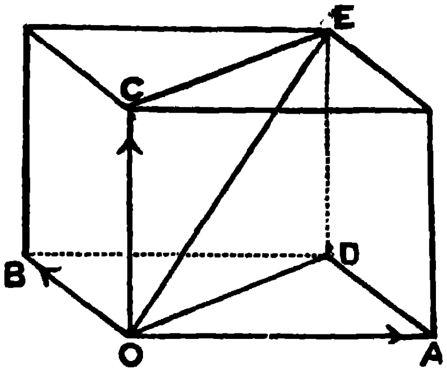
This proposition is also true when the velocities are not all in the same plane.

24. The **converse of the Polygon of Velocities** is not necessarily true: that is, 'If a point be at rest when possessed of any number of velocities more than three, we cannot represent these velocities in magnitude by the sides of *any* polygon drawn parallel to their directions,' because it does not follow that rectilineal figures (except triangles, Euc. vi. 4) which are equiangular must have their sides proportional—in other words, they need not be *similar* figures.

THE PARALLELOPIPED OF VELOCITIES.

25. Statement.—*If a point have three Component Velocities which are represented in magnitude and direction by the*

three edges of a parallelopiped, their resultant is represented in magnitude and direction by the diagonal through the point.



Let the component velocities be represented in magnitude and direction by *OA*, *OB*, *OC*, the adjacent edges of a parallelopiped. Their resultant is represented in magnitude and direction by the diagonal *OE*.

Draw *OD*, *CE*, and *OE*.

The resultant of the components *OA* and *OB* will be represented by *OD*. (Art. 18.)

And since *ODEC* is a parallelogram,

the resultant of *OD* and *OC* will be represented by *OE*;

$\therefore$ *OE* will represent the Resultant of *OA*, *OB*, *OC*. (Q.E.D.)

26. If these Component Velocities act in directions at right angles to each other, we can very easily obtain an expression for the resultant in terms of the components.

Let v be the resultant of the components v_1, v_2, v_3, in the directions *OA*, *OB*, *OC* respectively,

$$\text{Then } OD^2 = v_1^2 + v_2^2, \text{ (Art. 19. Cor. i.)}$$

$$\text{and } OE^2 = OD^2 + v_3^2;$$

$$\therefore\ OE^2 = v_1^2 + v_2^2 + v_3^2;\ \therefore\ v^2 = v_1^2 + v_2^2 + v_3^2.$$

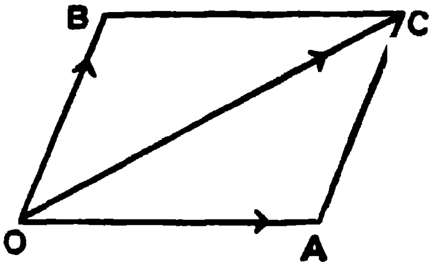
Example.—Three velocities 16, 63, and 156, are impressed on a point at right angles to each other; required the magnitude of their resultant.

$$\begin{aligned} v^2 &= 16^2 + 63^2 + 156^2 \\ &= 256 + 3969 + 24336 \\ &= 28561; \\ \therefore\ v &= 169. \end{aligned}$$

Therefore the resultant velocity is 169.

27. Before noticing the general method of finding the Resultant of several velocities in the same plane, it is necessary to point out the way in which a velocity may be resolved into two velocities at right angles to each other.

28. As we have seen that two Simultaneous Velocities can be combined into a resultant velocity (Art. 18), so we can resolve a single velocity into two simultaneous component velocities.



For example, if a velocity be represented by OC, then it can be resolved into the simultaneous components OA and OB.

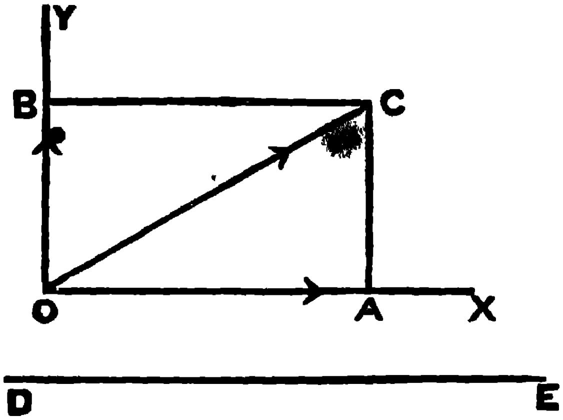
Now in resolving a velocity into two others we cannot speak of one component as quite independent of the other, because by the geometry of the parallelogram the *magnitude* of OA will evidently depend on the *direction* of OB. Hence, when we speak of *the* component of a velocity in any particular direction, we must know the direction of the other component. In practice it is most convenient to take the second direction at right angles to the first; and we may then speak of the *component of a given velocity in any direction*, it being understood that the direction of the other component is perpendicular to it.

The component in this case is called the '*resolved part*' of the velocity in the required direction.

29. *To determine the components of a velocity in any given direction and at right angles to that direction.*

Let OC denote any velocity, and it is required to find the components of which it is the resultant, one in the direction of the straight line DE, the other at right angles to this direction.

Through O draw OX parallel to DE, and OY at right angles to OX.



Draw CA and CB perpendicular to OX and OY respectively.

Then $OACB$ is a parallelogram, and OC will be the resultant of the velocities represented by OA and OB.

Since these velocities are at right angles to each other, it follows that the *resolved part* of OC in the direction OX (or DE) will be represented by OA $(=v_1)$, and in the direction OY by OB $(=v_2)$. Let the angle $AOC=\theta$.

$$\text{Now}\quad OA=OC\cos AOC,$$
$$\text{and}\quad OB=OC\cos(90^\circ-AOC);$$
$$\therefore v_1=v\cos\theta$$
$$\text{and}\quad v_2=v\cos(90^\circ-\theta)=v\sin\theta.$$

Therefore *the component of a velocity in any direction is found by multiplying the velocity by the cosine of the angle between the direction of the velocity and the line along which the component is required to act.*

NOTE 1.—If two velocities are inclined at any angle, they have *only one* resultant; and this may always be found by the Parallelogram of Velocities. But any velocity may be decomposed into *any number* of pairs of components, because by Geometry the same straight line may be a diagonal of an infinite number of parallelograms; or on any straight line an infinite number of polygons may be constructed. (*See*

Art. 22, Cor.) If, however, one component be given in direction, its magnitude, and that of the other at right angles to it, may always be found.

NOTE 2.—If in the diagram OC denote a velocity in a direction between north and east, then while the point has moved from O to C, it is evident that it has moved *due north*, a distance represented by OB, and *due east*, a distance represented by OA.

Example i.—If a point move with a velocity of 100 in a direction making an angle of 60° with the horizon, find its horizontal and vertical components.

$$\text{Hor. comp.} = 100 \cos 60^\circ = 100 \times \tfrac{1}{2} = 50.$$

$$\text{Vert. comp.} = 100 \cos 30^\circ = 100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3} = 86{\cdot}6.$$

Example ii.—A point moves with a velocity of 1000 f.-s. in a direction inclined to the horizon at an angle of 45°; find its vertical component.

Let v_2 denote the vertical component.

$$\text{Then } v_2 = 1000 \cos 45^\circ = 1000 \frac{1}{\sqrt{2}} = 707{\cdot}1 \text{ f.-s.}$$

EXAMPLES—VI.

1. Find the horizontal and vertical components of a velocity of 80 whose direction is inclined at 30° to horizon.

2. A velocity of $200\sqrt{3}$ f.-s. is inclined at an angle of 30° to the horizon, find its horizontal component.

3. A particle moves with a velocity of 120 up a plane, inclined at an angle of 30° to the horizon; with what speed is it travelling horizontally and vertically?

4. A ship is making 14 miles an hour on a north-east course; how fast is she moving east, and also north?

5. Three velocities $20\sqrt{3}$, $30\sqrt{2}$, 40, are impressed on a point in directions making angles of 30°, 45°, and 60° respectively with a fixed straight line; find the velocity of the point in a direction parallel to that line.

6. Three velocities, 12, 15, 24, are inclined at angles 30°, 45°, 120° respectively to a given straight line; find the velocity with which the point tends to move along that line, and also along the line at right angles to it.

7. Two velocities, v_1 and v_2, in the direction of OA and OB respec-

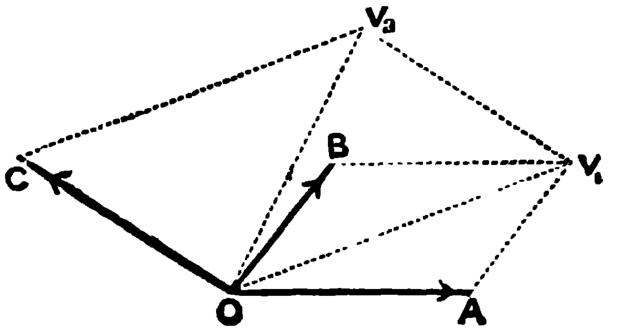
tively, include an angle of 30°; find their combined components along a line which makes an angle of 120° with the direction of OA, and 30° with the direction of OB.

30. *To find the Resultant of any number of velocities possessed by a point.*

This may be done in three ways.

(i.) *By repeated applications of the Parallelogram of Velocities.*

Find the resultant of any two of the velocities; then of this resultant and the next velocity; and so on for any given number of components. The final resultant so found will be the Resultant of the given velocities.



Let OA, OB, OC represent three velocities.

Completing the parallelograms as in the figure, we have,

OV_1 the resultant of OA and OB, (Art. 18)

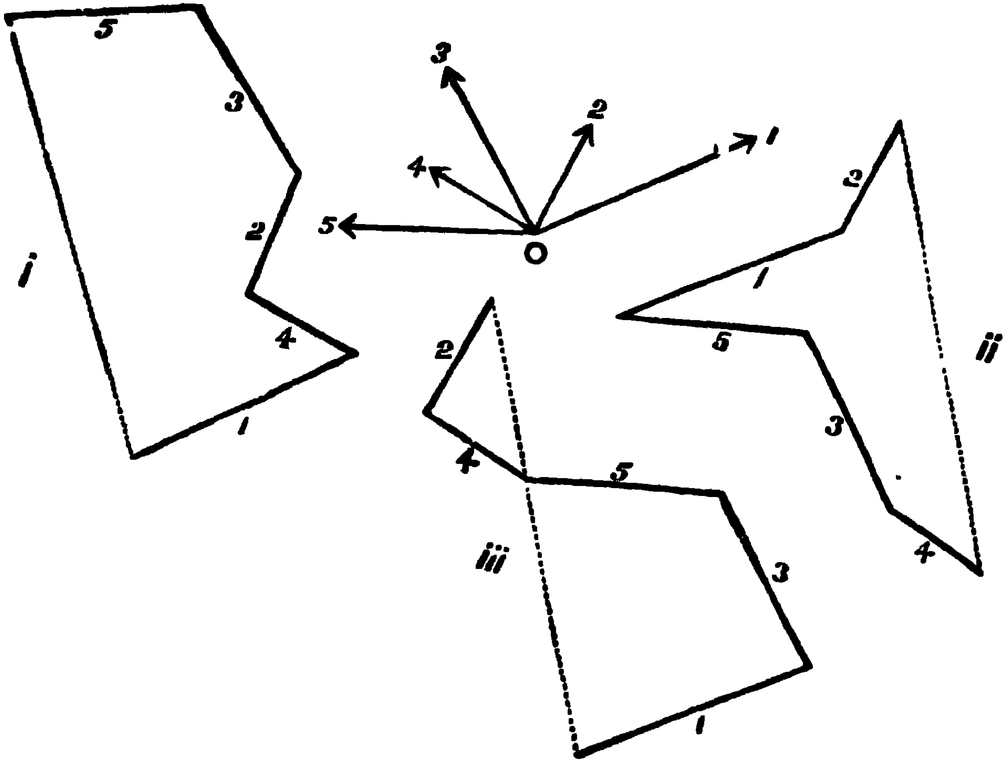
and OV_2 the resultant of OV_1 and OC;

$\therefore$ of OA, OB, OC.

(ii.) *By the Polygon of Velocities.*

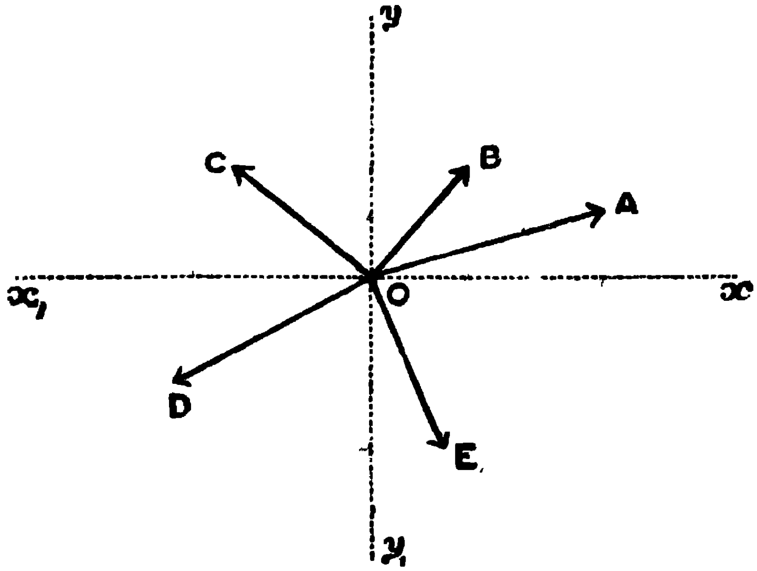
Construct a figure whose sides taken in order will represent the magnitude and direction of the given components; then the line closing the figure and *taken in the opposite direction* will completely represent the Resultant. (Art 23.)

The student ought to notice that the order in which we draw the lines is quite immaterial, and that it will not make any difference if the lines cross each other. The annexed figures will make this clear.



In Fig. i. the lines are drawn in the order 1, 4, 2, 3, 5.
In Fig. ii. „ „ „ 4, 3, 5, 1, 2.
In Fig. iii. „ „ „ 1, 3, 5, 4, 2.

In each figure the dotted line represents the Resultant in magnitude and direction.



(iii.) *By resolving the velocities along two straight lines at right angles to each other, and compounding the components thus found.*

Let *O* be a point possessing velocities fully represented by *OA*, *OB*, *OC*, *OD*, *OE*.

Through O draw two straight lines xx_1 and yy_1 at right angles to each other.

Resolve the velocities along xx_1 and yy_1 respectively. The components along xx_1 in the direction Ox will then be
$OA \cos xOA$; $OB \cos xOB$; $OC \cos xOC$ . . . . (Art. 29.)

The components along yy_1 in the direction Oy will be

$OA \cos yOA$; $OB \cos yOB$; $OC \cos yOC$ . . . .

or, $OA \sin xOA$; $OB \sin xOB$; $OC \sin xOC$ . . . .

Let X denote the *algebraical* sum of the components in the direction Ox. (Art. 17, Note.)*

Let Y denote the *algebraical* sum of the components in the direction Oy.

$$\therefore X = OA \cos xOA + OB \cos xOB + \ldots.$$

And $Y = OA \sin xOA + OB \sin xOB + \ldots.$

Then we have two velocities, X and Y, at right angles to each other. Let V denote the resultant of these velocities.

Then $V^2 = X^2 + Y^2$. (Art. 19, Cor. i.)

This determines the *magnitude* of the Resultant.

If θ be the angle which V makes with Ox, then

$$\tan\theta = \frac{Y}{X}.$$

And this determines the *direction* of the Resultant.

NOTE I.—The usual conventions about the positive and negative directions of straight lines must be carefully attended to. For example, if $X=8$ and $Y=-6$, then $V=10$, and the point would move in the region xOy_1; but if $X=-8$ and $Y=-6$, then the point would move in the region x_1Oy_1 with the velocity $=10$.

NOTE 2.—In practice it is often desirable, in order to lessen the labour of numerical computation, to draw xx_1 or yy_1 coincident with the direction of one of the component velocities.

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* The student will remember that a line Ox_1 is *positive* when measured in the direction Ox_1, but it must be considered *negative* when measured in the direction Ox.

NOTE 3.—If the point is at rest, then $V=0$.

$\therefore X^2+Y^2=0$; and $\therefore X$ must $=0$ and Y must $=0$ separately.

From this we infer that if a point be at rest when possessed of any number of velocities, then the algebraical sums of their components in any two directions at right angles to each other must be separately equal to zero; and conversely, if these algebraical sums be each equal to zero, the point is at rest.

Example.—A point has four simultaneous velocities of 10 f.-s. The angle between first and second is 30°, between the second and third is 60°, between the third and fourth is 30°. Find the magnitude of the resultant.

Drawing xx_1 to coincide with the first component, it is evident that yy_1 will coincide with the third.

Then $X=10+10\cos 30°+10\cos 90°+10\cos 120°$,

and $Y=10\cos 90°+10\cos 60°+10+10\cos 30°$;

$$\therefore X=10+\frac{10\sqrt{3}}{2}+0-10.\tfrac{1}{2}=5+5\sqrt{3},$$

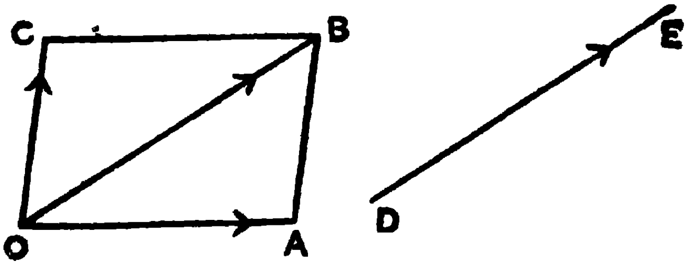
$$Y=0+10.\tfrac{1}{2}+10+\frac{10\sqrt{3}}{2}=15+5\sqrt{3},$$

$$V^2=X^2+Y^2=(5+5\sqrt{3})^2+(15+5\sqrt{3})^2=400+200\sqrt{3}$$
$$=100\,(4+2\sqrt{3});$$

$$\therefore V=10\,(\sqrt{3}+1)=27\cdot 32 \text{ f.-s.}$$

CHANGE OF VELOCITY.

31. When a certain velocity has been changed into another velocity the question may be asked, 'What has been the change of velocity?' We can answer this by saying that a given component and an unknown velocity have produced a given resultant, and *the unknown component represents in magnitude and direction the 'change of velocity.'*



32. To make this clear, suppose that a velocity *OA* at

any instant is changed into a velocity DE at another instant, and we require to find the change of velocity in the interval. We draw OB equal and parallel to DE, join AB, and complete the parallelogram $OABC$. Then OB is the resultant of OA and OC.

That is, OC (or AB) in combination with OA has produced a velocity represented in magnitude and direction by DE.

Therefore OC (or AB) measures the ‘change of velocity’ in the interval in both magnitude and direction.

Note.—The student must therefore remember that the ‘change of velocity’ is not denoted by the difference between the velocities at the beginning and end of any interval unless the direction of the velocity has during that interval been in the same straight line. This is important.

Example i.—A velocity of 10 to the north is changed into a velocity of 6 to the south; find the change of velocity.

Here we ask, ‘What velocity when compounded with a velocity of 10 north will produce a velocity of 6 south?’

Let v=velocity required.

Then if a velocity to the north is positive, one to the south will be negative.

Therefore, $v+10=-6$.

$\therefore\ v=-16$.

Hence the change of velocity is 16 to the south.

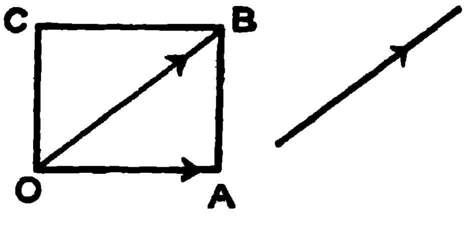
Example ii.—A velocity of 6 to the east is changed to a velocity of $6\sqrt{2}$ to the north-east; find the change of velocity.

Construct the fig. as in Art. 32.

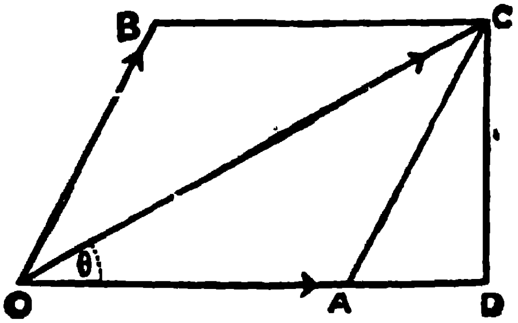
The $\Delta\ OAB$ is evidently right-angled at A, and $AOB=45°$,

$\therefore\ AB=AO=6$.

Hence the change of velocity is 6 to the north.



Example iii.—A point is moving with a velocity of 10 (OA), and during an interval its change of velocity is 14 (OB), in a direction making an angle of 60° with its first direction. Find the motion of the point at the end of the interval.



OC is the resultant of OA and OB.

$$OC^2 = 10^2 + 14^2 + 2 \,.\, 10 \,.\, 14 \,.\, \cos 60^\circ.$$
$$\therefore OC^2 = 100 + 196 + 140;$$
$$= 436;$$
$$\therefore OC = 2\sqrt{109}.$$

$$\tan\theta = \frac{CD}{OD} = \frac{\text{Vertical comp. of } OB}{OA + \text{Hor. comp. of } OB} = \frac{14 \sin 60^\circ}{10 + 14 \cos 60^\circ} = \frac{7\sqrt{3}}{10+7}.$$

The point is moving with a velocity of $2\sqrt{109}$ in a direction which makes an angle with its first direction whose tangent is $\frac{7\sqrt{3}}{17}$.

EXAMPLES—VII.

1. A point has a velocity at a certain instant of 10 f.-s. to the west, and at another instant a velocity of 6 f.-s. to the north-west; find the change of velocity in the interval.

2. The direction of the velocity (v) of a point is changed by 60°, but its magnitude remains the same; find the change of velocity in the interval.

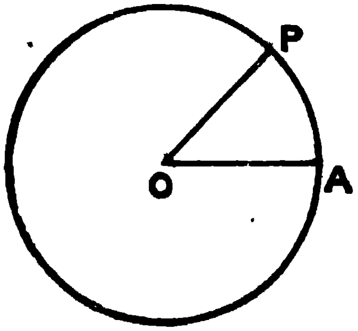
3. A point has a velocity of 12 to the east at one instant and a velocity of 12 to the south at another instant; what is the change of velocity?

4. A point is describing a circle whose radius is 10 feet with a velocity of 25 f.-s.; find the change of velocity during the time the point describes an arc (1) of 60°; and (2) of 90°.

5. A point has velocity of $\frac{1}{4}$ f.-s., and it suffers a change of velocity of $\frac{3}{10}$ f.-s. in a direction which is inclined at an angle 60° to the original direction of the motion; find the final velocity.

ANGULAR VELOCITY.

33. If O be a fixed point, P any other point, and OA a fixed straight line, then the rate at which the angle AOP is altering measures the **angular velocity** of P about O. It is evident that if P be not moving in a straight line to or from O, it must always have *some* angular velocity about O.



34. The **Unit of angular velocity** is that uniform angular velocity with which a point describes the unit angle (a radian) in unit time.

The symbol ω is used to denote the angular velocity of a point expressed in circular measure.

When equal angles are described in all equal intervals of time the angular velocity is *uniform;* if not, it is said to be *variable.*

Uniform angular velocity is measured by the number of unit angles described in unit of time.

Variable angular velocity is measured at any instant by the number of unit angles which would be described in a unit of time if the angular velocity at that instant remained uniform for the unit of time.

If ω be the angular velocity of a point P, and t the time in which it describes the angle AOP, then $AOP = \omega t$.

35. If v be the *linear* velocity with which P is moving in the circumference of the circle whose radius is r, then we may connect the linear and angular velocities as follows:—

Let motion take place for t seconds.

Then $AP = vt$, and $AOP = \omega t$:

but $AOP = \frac{AP}{OA} = \frac{vt}{r}$;

$$\therefore \omega t = \frac{vt}{r};$$

$$\therefore \omega = \frac{v}{r},$$

i.e. the measure of the angular velocity $= \frac{\text{linear velocity}}{\text{radius of the circle}}$.

EXAMPLES—VIII.

Take $\pi = \frac{22}{7}$.

1. If a point be moving on the circumference of a circle whose radius is 3 feet with a velocity of $\frac{22}{21}$ f.-s., find its angular velocity.

$$\omega = \frac{\frac{22}{21}}{3} = \frac{22}{63}.$$

2. A point is moving in a circle of 20 feet with a velocity of 20 feet per second; find the angular velocity.

3. A point is moving with an angular velocity of $\frac{\pi}{3}$ in a circle whose radius is 10 feet; find its linear velocity.

4. A point is moving with uniform velocity v along the circumference of a circle, radius r; find its angular velocity about a point on the circumference at the other end of the diameter.

5. If a ball at the end of a string make a complete revolution in one second, find the angular velocity.

6. Compare the angular velocities of the hour-hand, the minute-hand, and the second-hand of a watch.

7. If the hour-hand be 3 inches long, find the linear velocity of its extremity.

If the minute-hand be twice as long as the second-hand, show that the end of the latter moves 30 times as fast as the end of the former.

8. If the earth's radius be 4000 miles, find the linear velocity of a point situated on the equator in foot-second units.

9. Compare the linear velocities of the extremities of the hands of a clock, the hour-hand being $2 \cdot 2$ inches long, and the minute-hand $3 \cdot 6$ inches.

10. If a point move with a velocity v in a circle whose radius is r, and after a time t has described an angle AOP, show that the velocities of the point in the directions OA and a line perpendicular to OA are $-v\sin\frac{vt}{r}$, and $v\cos\frac{vt}{r}$, respectively.

RELATIVE VELOCITY.

36. DEF.—When the straight line joining two points changes either in magnitude or in direction, or in both, the points are said to have **Relative Velocity**.

37. Thus, if two points are moving in the same straight line with different velocities, the line which joins them is changing in magnitude but not in direction, and each of the points has a velocity relative to the other.

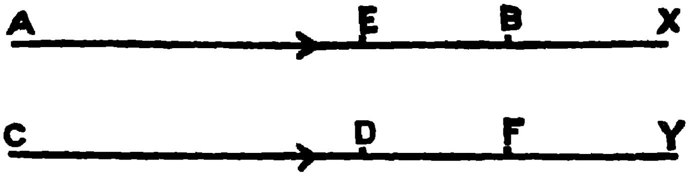
If a point is moving on the circumference of a circle its distance from the centre changes in direction but not in magnitude, and the moving point has a velocity relative to the centre.

Generally, if two points move with different velocities along two lines, the line which joins them changes both in magnitude and direction, and each of these points has a velocity relative to the other.

38. Now suppose that two trains are moving in the same direction with equal velocities (however great) on parallel lines close together, then a person sitting well back in a carriage of either train and in full view of the other, *would not be conscious of that other train possessing any motion whatever*,—supposing, of course, the motion of his own train to take place on perfectly smooth rails.

If the velocities of the trains are not equal, then a person in either would be conscious of a velocity in the other equal to the difference of the velocities, but this velocity would appear to have opposite directions to the respective passengers.

39. This may be very easily explained by a figure:—



Let one train move along AX with a velocity AB;
and the other along CY with a velocity CD.
Cut off from AB a part $AE=CD$.
Then $AB=AE+EB$,

i.e. the velocity of the faster is equal to the resultant of two components one of which is equal to the velocity of the slower train.

Now the component AE of the velocity AB would not be apparent to a person moving with the equal velocity CD. There remains the component EB, which will carry the faster train ahead of the other; and this component is the difference of the velocities. Thus to a person in the slower train, the faster will have only a velocity represented by the difference of the velocities in the direction of the faster.

Next, in CY take $CF=AB$.
Then $CD=CF-DF$, or $=CF+FD$. (Art. 15.)

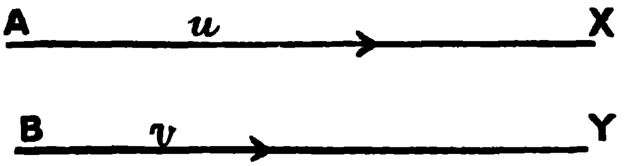
i.e. CD is equal to a velocity AB in its own direction, and a velocity FD in the opposite direction.

Now a person in the faster would not be conscious of the component CF, but he would be conscious of the component FD, and of that only. And hence the slower train would appear to him *to move backwards* with the velocity FD, which is the difference of the velocities.

In each case, therefore, where the velocities have the same direction, the magnitude of the Relative Velocity is the *difference of the velocities*.

The student can easily show in like manner that when the velocities have opposite directions the magnitude of the Relative Velocity is equal to their *sum.*

40. The same result may be more briefly shown as follows, but the principle is the same:—



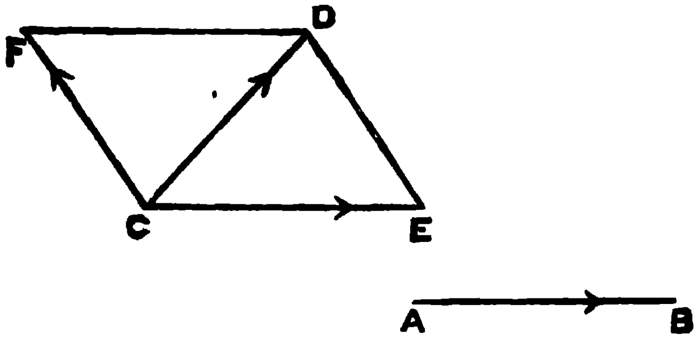
Let the velocity of the faster train $A, = u$; that of the slower $B, = v$. Suppose that $u = v + x$. A person in B is not conscious of the component v in A, but he is of x; and this therefore will be the velocity with which the faster moves *ahead* of the other.

Again $v = u - x$.

A person in A is only conscious of the component $-x$ in B, and this will appear to carry the slower train *backwards* at a rate x.

In each case x the difference of u and v will be the *magnitude* of the Relative Velocity, and its *direction* will depend upon which train is selected as that whose relative motion is wanted.

41. Let us next consider the case *where the directions are not parallel.*



Suppose that two ships close to one another at A and C have velocities represented in magnitude and direction by AB and CD respectively.

Resolve CD into two components, one of which, CE, is *equal and parallel* to AB, and complete the parallelogram $CEDF$. (Art. 28.) Then CD is the resultant velocity of velocities represented by CE and CF.

Now by what has gone before a person in A is not conscious of that part of *C's* motion represented by *CE* (Art. 38), but he is of that motion only which is represented by *CF*. Hence *CF* represents the Relative Velocity of *C* with regard to *A* in magnitude and direction.

In the same way the velocity of *A* with regard to *C* may be found. It is equal to the other in magnitude, but opposite in sign. The student will find it profitable to draw the figure as a test of his grasp of this part of the subject.

NOTE.—The shortest distance apart between the ships may be found by drawing from *A* a perpendicular to *FC*, produced if necessary. The Relative Velocity is often denoted by *V*.

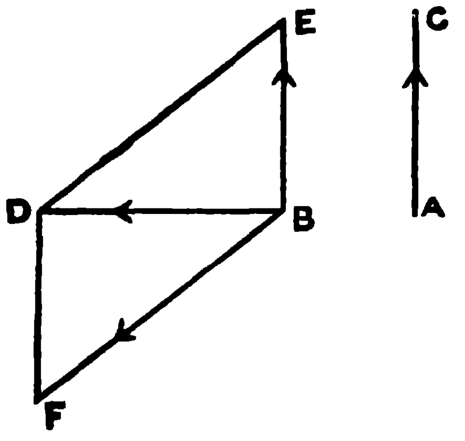
Example i.—One point is going due east with a velocity of 10, another due east with a velocity of 25; find the velocity of the latter relative to the former.

$$V = 25 - 10 = 15 \text{ due east.}$$

Example ii.—If the latter point were travelling west with a velocity of 12, what would be the relative velocity?

$$V = 10 + 12 = 22 \text{ due west.}$$

Example iii.—One train *A* is going N. at the rate of 30 miles an hour, another *B* due W. at 40 miles an hour; find the velocity of the latter relative to the former.



We break up *BD* (40) into *BE* (30) due north, and *BF* (*V*) equal and parallel to *ED*, somewhere between south and west.

$$V^2 = BD^2 + DF^2 = 40^2 + 30^2 = 2500$$

$$\therefore V = 50;$$

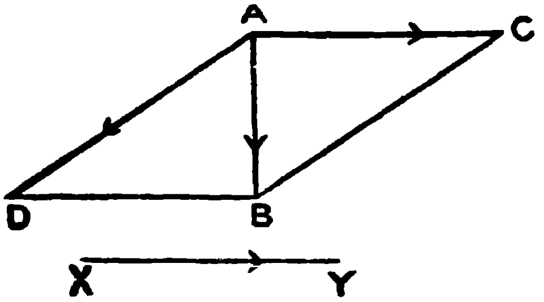
$$\tan DBF = \frac{DF}{DB} = \frac{3}{4};$$

$$\therefore DBF = \tan^{-1}\left(\tfrac{3}{4}\right).$$

Hence the magnitude of the relative velocity is 50 miles an hour, and its direction makes with the east and west line an angle whose tangent is $\frac{3}{4}$.

Example iv.—Explain why rain falling vertically appears to a person in a train moving rapidly through it to move in a slanting direction.

Let a drop of rain have a vertical velocity *AB*, and let *XY* represent the velocity of the train. Resolve *AB* into the components *AC* equal and parallel to *XY* and *AD* (equal and parallel to *CB*). Then *AD* will be the only component of which the traveller can take note, and *BAD* will be the angle of apparent deflection from the vertical.



42. Now if two points are in motion their *relative* motion is not changed if the same velocity, in magnitude and direction, is impressed on each; and hence another very useful method of finding the Relative Velocity of a point *A* with regard to another *B* is to imagine impressed upon each, in composition with its own velocity, a velocity equal and opposite to the velocity of *B*. *B* will now be at rest, and the new velocity of *A* will be its relative velocity with regard to *B*. This method is often more easy of application than that of resolving one of the velocities.

Thus in **Example iii.** above, if we impress on both *A* and *B* a velocity of 30 miles an hour in a southward direction, the train *A* will be at rest, and the diagonal of the parallelogram whose sides are *BD* and *BE* (drawn due south from *B*) will be *BF* as before.

EXAMPLES—IX.

1. Two points are moving along a line in the same direction with velocities of 30 and 20; find their relative velocity.

2. If the points in Ex. 1. were moving in opposite directions, find their relative velocity.

3. If the points in Ex. 1. were moving along lines at right angles to each other, find their relative velocity.

4. Two trains run through a station on lines whose directions include an angle of 60° with velocities of 20 and 35 respectively; find their relative velocity.

5. A ship is sailing due N. with a speed of 10 knots, and another S.W. with a speed of 15 knots; find the velocity of the latter relative to the former.

6. Two points *A* and *B* are moving on a horizontal plane, *A* with a velocity of 2 f.-s. due N., and *B* with a velocity of $\sqrt{3}$ f.-s. due E.; find the velocity of *B* relative to *A*.

7. Two steamers leave the mouth of a harbour at the same time, one going due E. with a speed of 12 knots, and the other due N. with a speed of 16 knots; show that they separate from each other at the rate of 20 knots.

8. A ship is sailing due N. at 5 miles an hour; in what apparent direction and with what relative velocity must a man run on her deck that his actual motion over the ground may be due W., and his absolute velocity 5 miles an hour?

9. A ship *A* is sailing due E. 6 knots. A person in *A* observes another, *B*, on the port bow; *B* appears to be going due N., and is known to be going 12 knots; find her course.

10. Why does a person walking rapidly in rain which is descending vertically hold his umbrella somewhat in front?

11. A man walks at the rate of $3\frac{3}{4}$ miles an hour in rain falling vertically at the rate of 40 f.-s.; in what direction will the rain strike him?

12. *A* is due W. of *B*, and distant from him 200 feet. *A* starts to walk due E. with a velocity of 8 f.-s., and at the same instant *B* proceeds due N. with a velocity of 6 f.-s.; find (1) their relative velocity, (2) their nearest distance to each other, (3) when they are nearest to each other.

13. Two persons are placed at *A* and *B*, 5 yards apart. The first starts to walk from *A* to *B* at the rate of 3 yards per second, and the other starts from *B* at right angles to the line *BA* at the rate of 4 yards per second; what is their shortest distance apart, and the time of reaching it?

14. A boy is running with a speed of *a* f.-s., and passing through rain falling vertically with a velocity of *b* f.-s. He holds an open tube in such a manner that a drop entering it traverses its length along the axis; at what angle does he hold it?

15. To a passenger in a train moving at the rate of 45 miles an hour, through rain falling vertically, the direction of the drops appears to make an angle $\tan^{-1}(\frac{3}{2})$ with the vertical; find the velocity of the rain.

16. A traveller moving due E. at 4 miles an hour observes that the wind seems to blow directly from the N., and on doubling his speed it appears to blow from the N.E.; find the true direction of the wind, and its velocity.

17. A farmer driving to market on a road going due W. feels the wind from the N.W.; and returning at the same speed he feels the wind N.E. Explain this, the wind not having changed its true direction during the day.

18. A wind is on the beam; as the ship's speed increases the wind appears to draw more ahead. Explain this by a diagram.

19. Two trains moving in the same direction on parallel lines appear to a passenger to pass each other in 2 minutes; on the faster reversing its direction, they repass each other in 30 seconds. The slower speed being 21 miles an hour; find the faster.

20. *A*, *B*, *C* run a race, *B* having 30 and *C* having 50 yards start. After 32 seconds *B* passes *C*, and 16 seconds later is passed by *A*. When did *A* pass *C*?

21. The wheel of a carriage is 4 feet in diameter, and it makes $3\frac{1}{2}$ revolutions per second; find the velocity of any point in the circumference relative to the centre.

22. Three railways, *AB*, *AC*, *AD*, which make angles of 30° with each other, meet in *A*, and three engines pass through *A* at the same time, on the three lines with uniform velocities, such that at any instant they are in a line perpendicular to *AB*. Show that to a person on the first engine the velocities of the other two engines would appear to be as 1 : 3.

23. A train is going 28 miles an hour, when a bullet moving horizontally enters a compartment at one corner and passes out at the corner diagonally opposite. The compartment is 8 feet between the windows, and is 6 feet wide, the *sine* of the angle between the paths of the bullet and train being $\frac{3}{5}$; find the velocity of the bullet.

24. Two particles move with the same uniform velocity v on the circumferences of two concentric circles (radii r and $2r$), starting from points on the same straight line *OAB* through the centre and on the same side of it. Determine the direction and magnitude of their relative velocity when the particle on the smaller circle has described one-third of the circumference.

25. If a watch be laid face upwards on a table, and moved without rotation along the line joining the figures 9 and 3 on the dial, with the velocity with which the extremity *P* of the minute hand moves when at rest, prove that the velocity of *P* at any instant relative to the table is perpendicular and proportional to the line joining P at that instant with the position of P at any half-hour.

26. If x be the distance between two points moving uniformly in one plane, V their relative velocity, and u, v the resolved parts of V in, and perpendicular to, the direction of x, show that their shortest distance apart is $\frac{xv}{V}$, and the time of arriving at this is $\frac{xu}{V^2}$.

27. Two railway trains are moving in lines at right angles towards a crossing, 1 mile and $\frac{1}{2}$ mile off respectively, the velocities being 40 and 30 miles per hour respectively. Show that their shortest distance apart is $\frac{1}{5}$ mile.

28. A ship A sails due N at $4\sqrt{3}$ miles per hour. Another ship B is at a given time 8 miles from A, bearing 30° west of north. Prove that if B sails at 4 miles per hour there are two directions in which she will meet A, one at the end of 1 hour, the other at the end of 2 hours.

29. Practice is carried out from a ship steaming at u f.-s. in a circle round a target abeam, with guns firing with a muzzle velocity of v f.-s. Prove that the axis of the gun must be trained aft through an angle $\sin^{-1}\left(\frac{u}{v}\right)$.

For instance, if $u = 15$ knots, and $v = 1500$, show that this angle becomes nearly 1°.

30. On board a ship steaming N.E. at 12 knots, the relative velocity of the wind as estimated from the clouds is 4 knots from N. Changing her course without altering speed, the wind appears to blow from N.W. Construct the new course of the ship.

31. When a steamer is going N. at 15 knots, a vane on the masthead points E.N.E., and when the steamer stops, the vane points S.E.; show that the velocity of the wind is also 15 knots.

Show that if the vane points due E. the course of the steamer has been changed to N.E.

CHAPTER II.

ACCELERATED VELOCITY.

43. Def.—When the velocity of a point is changing, the '*rate at which the velocity is being changed*' is called the **Acceleration of the Velocity.**

44. The term *Acceleration* is here used to denote any change in the velocity, whether that change be an increase or a decrease. If the acceleration measured in one direction be regarded as positive, then when measured in the opposite direction it must be considered as negative; and a *negative acceleration* is what is called a *retardation* in popular language.

45. The phrase 'acceleration of a point' is often used instead of 'acceleration of the velocity of a point.'

46. A velocity involves both *magnitude* and *direction*, and if either, or both, of these elements be changing, the velocity is said to be 'accelerated.'

For instance, the velocity of a falling body is said to be accelerated, because, while the direction of the velocity remains the same, the *magnitude* of the velocity is being changed. Again, the velocity of a point which describes the circumference of a circle with uniform velocity is accelerated, because, while the magnitude of the velocity remains the same, the *direction* of the velocity is being changed.

The student must bear in mind this two-fold capability of change possessed by a velocity. The acceleration arising from the change in direction only is sometimes forgotten, or not understood.

47. The 'rate of change in the velocity' of a point may be either uniform or variable.

48. Def.—**Uniform Acceleration** in any direction means that the 'rate of change of velocity' in that direction is the same at every instant.

49. Def.—**Variable Acceleration** means that the 'rate of change of velocity' is not the same at any two consecutive instants.

50. The **Unit Acceleration** is the 'unit of velocity added per unit of time.'

The British Unit Acceleration, therefore, is 'a velocity of a foot per second added per second.'

The Unit Acceleration in the C. G. S. system is 'a velocity of a centimetre per second added per second.'

And generally: If 'x feet' and 't seconds' are the units of space and time selected, then the Unit Acceleration will be 'a velocity of x feet per t seconds added per t seconds.'

51. **Uniform Acceleration is measured** by the rate at which the velocity is increased *per unit of time.*

Thus an acceleration of 12 in the foot-second units will imply that the velocity of a moving point is increased by 12 feet per second every second. And this will be denoted by '12 f.-s.-s.'

A point moving from rest with an acceleration of 5 f.-s.-s. will therefore have velocities of 5, 10, 15, 20, ... feet per second at the end of 1, 2, 3, 4, ... seconds respectively; or, velocities of $2\frac{1}{2}$, 5, $7\frac{1}{2}$, 10, ... feet per sec. at the end of $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, ... seconds respectively.

And if after successive seconds we discover that a moving point has velocities of 20, 16, 12, ... feet per second, we are justified in saying that the point has been moving with a *negative* acceleration of 4 feet per second per second. (Art. 44.)

52. **Variable acceleration is measured** at any instant by the increase of velocity which would be added in a unit of time if the rate of increase of velocity at that instant remained the same throughout the unit of time.

53. DEF.—The **average** or **mean acceleration** during any time may be found by dividing the whole velocity gained during the time by that time.

It may be added here, once for all, that when we speak of dividing velocity by time, we divide the numerical measure of the one by that of the other. The phrase otherwise is nonsense.

54. NOTE 1.—Acceleration is now generally denoted by the symbol 'a.' In older works on Mechanics the letter 'f' was used, when the erroneous term 'accelerating force' was employed to express an increase of velocity.

NOTE 2.—If nothing be said to the contrary, the student will understand that we are using the 'foot-second' system of units.

55. PROPOSITION I.—*If we pass from the 'foot-second' units to any other units of space and time, then the New Unit of Acceleration varies directly as the new unit of space, and inversely as the square of the new unit of time.*

Let x feet be the new unit of space, and let t seconds be the new unit of time.

By Definition (Art. 50) the New Unit of acceleration

$\equiv$ a vel. of x feet per t secs. added per t sec.

$\equiv$ x times a vel. of 1 foot per t secs. added per t secs.

$\equiv \dfrac{x}{t}$ times a vel. of 1 foot per 1 sec. added per t secs.

$\equiv \dfrac{x}{t^2}$ times a vel. of 1 foot per 1 sec. added per 1 sec.

$\equiv \dfrac{x}{t^2}$ times the 'foot-second' unit of acceleration.

$\therefore$ New Unit $= \frac{x}{t^2} \times$ Old Unit

$= \frac{k.x}{t^2}$, because the Old Unit is constant;

$\therefore$ New Unit $\propto \frac{x}{t^2}$, by Algebra,

i.e. New Unit $\propto \frac{\text{Unit of space}}{(\text{Unit of time})^2}$ (Q.E.D.)

56. PROPOSITION II.—*The Measure of an acceleration, when we pass from the 'foot-second' units to any other units, varies directly as the square of the unit of time, and inversely as the new unit of space.*

Since any Quantity = Measure × Unit of that quantity, we can assert, by Algebra, that the *Measure* of the quantity *varies inversely* as the Unit employed.

And therefore, by Proposition I., we infer at once that the Measure of an acceleration $\propto \frac{(\text{Unit of time})^2}{\text{Unit of space}}$. (Q.E.D.)

Or, the Proposition may be established directly from Definition, as follows:—

Let x feet be the new unit of space,
and let t seconds be the new unit of time.
Let a be the measure of an acceleration in the 'foot-second' units.
Let a_1 be the measure of the *same* acceleration in the 'x feet-t seconds' units.

Then a means that the velocity is increased by
a times a vel. of 1 foot per 1 sec. per 1 sec. (Art. 51.)

or, $\frac{a}{x}$ times a vel. of x feet ,, ,,

or, $\frac{at}{x}$ times a vel. of x feet per t secs. ,, or, $\frac{at^2}{x}$ times a vel. of x feet per t secs. per t secs.

$\therefore$, $\frac{at^2}{x}$ is the Measure of the acceleration in the new units.

But, by data, the Measure in the new units is a_1;

$$\therefore\ a_1 = \frac{at^2}{x}.$$

But a is a constant, $\therefore\ a_1 = \frac{kt^2}{x}$

$$\therefore\ a_1 \propto \frac{t^2}{x}$$

i.e. the Measure of the acceleration $\propto \frac{(\text{Unit of time})^2}{\text{Unit of space}}$. (Q.E.D.)

57. When required to change the measure of an acceleration from one system of units to another, the student is strongly advised to avoid stated formulæ, and to work each question directly from Definition. The following examples will sufficiently explain the method of working here recommended :—

Example i.—If 32 be the measure of an acceleration in the f.-s. system, find its measure in the 'foot-minute' system.

Let a be the measure required.

A velocity of 32 feet per second added per second

≡ ,, 32×60 feet per 60 seconds added per second

≡ ,, 32×60^2 feet per 60 seconds added per 60 seconds;

$\therefore\ a = 32 \times 60^2 = 115{,}200$ feet per minute per minute.

Example ii.—If 110 be the measure of an acceleration in the f.-s. system, find its measure in the 'mile-hour' system.

Let a be the measure required.

A velocity of 110 feet per second per second

≡ ,, $\frac{110}{5280}$ miles per second per second

≡ ,, $\frac{110 \times 3600}{5280}$ miles per hour per second

≡ ,, $\frac{110 \times 3600^2}{5280}$ miles per hour per hour;

$$\therefore\ a = \frac{110 \times 3600 \times 3600}{5280} = 270{,}000 \text{ miles per hour per hour.}$$

Example iii.—An acceleration which is measured by 32 in the f.-s. system is measured by 640 when the unit of space is 5 feet; find the unit of time.

Let t seconds be the unit of time.

Then 640 means that the velocity is increased by

640 times 5 feet per t seconds per t seconds;

or, 640×5 times 1 foot per t seconds per t seconds;

or, $\frac{640 \times 5}{t}$ times 1 foot per 1 second per t seconds;

or, $\frac{640 \times 5}{t^2}$ times 1 foot per 1 second per 1 second.

But, by data, 32 means that the same velocity is increased by 32 times 1 foot per 1 second per 1 second.

$$\therefore\ 32 = \frac{640 \times 5}{t^2}$$

$$\therefore\ t^2 = 100$$

$$\therefore\ t = 10.$$

The *unit of time* is therefore 10 *seconds*.

Example iv.—If a 10-acre field be represented by 100, and an acceleration by $58\frac{2}{3}$, which in the 'f.-s.' system is measured by 32; find the unit of time.

10 acres $= 4840 \times 10$ square yards;

$\therefore$ 100 square units $\equiv$ 48,400 square yards;

$\therefore$ 1 square unit $\equiv$ 484 square yards;

$\therefore$ 1 linear unit $\equiv$ 22 yards;

$\therefore$ Unit of space $=$ 66 feet.

Let t seconds be the unit of time.

A velocity of 32 feet per second per second

$\equiv$,, $\frac{32}{66}$ times 66 feet per second per second.

$\equiv$,, $\frac{32t}{66}$ times 66 feet per t seconds per second.

$\equiv$,, $\frac{32t^2}{66}$ times 66 feet per t seconds per t seconds.

and, by data, $58\frac{2}{3}$ means a velocity of $58\frac{2}{3}$ times 66 feet per t seconds per t seconds.

$$\therefore\ \frac{32t^2}{66} = 58\tfrac{2}{3}$$

$$\therefore\ t^2 = 121$$

$$\therefore\ t = 11.$$

The *unit of time* is therefore 11 *seconds*.

Example v.—If the unit of velocity be 6 f.-s. and unit of acceleration be 24 f.-s.-s.; required (1) the unit of time, and (2) the unit of space.

Let t secs. = unit of time, and x feet = unit of space.

By Definition, Unit Velocity = x feet per t seconds.

$$= \frac{x}{t} \text{ f.-s.}$$

By data, Unit Velocity = 6 f.-s.

$$\therefore \frac{x}{t} = 6 \quad . \; . \; . \; . \; . \; . \; . \quad (1.)$$

Again, by Definition, Unit Acceleration = unit velocity added per unit of time.

$$= \frac{x}{t} \text{ f.-s. added per } t \text{ seconds}$$

$$= \frac{x}{t^2} \text{ f.-s.-s.}$$

but, by data, Unit Acceleration = 24 f.-s.-s.

$$\therefore \frac{x}{t^2} = 24 \quad . \; . \; . \; . \; . \; . \; . \quad (2.)$$

Solving these equations, $t = \frac{1}{4}$ second, and $x = 1\frac{1}{2}$ feet.

Therefore, Unit of time = $\frac{1}{4}$ second; Unit of space = $1\frac{1}{2}$ feet.

EXAMPLES—X.

1. If the acceleration of a falling stone be 32 in the foot-second units, find its measure

    (*a*) when a yard and a second are the units;
    (*b*) when a yard and a minute are the units;
    (*c*) when a mile and a minute are the units;
    (*d*) when 5 yards and 10 seconds are the units;
    (*e*) when a fathom and $\frac{1}{2}$ second are the units;
    (*f*) when x feet and y seconds are the units;
    (*g*) when a mile and an hour are the units.

2. Find the ratio between an acceleration of 5 in the foot-second units, and an acceleration of 20 in the yard-minute units.

3. If 10 be the measure of an acceleration of 40 in the foot-second units, what number will represent an acceleration of 5 when the units are a yard and a minute?

4. If an acceleration is represented by 32 in the f.-s. system, and by 576 in a system of 'x feet, $6\sqrt{3}$ seconds' units; find x.

5. If an acceleration is denoted by 27 when referred to a quarter of a minute and 25 feet as units, by what number will it be expressed when referred to a foot and a second?

6. If the measure of an acceleration be the same in two systems, in which the units of time are 3 secs. and 5 secs., find the ratio of the units of space.

7. If a be the measure of an acceleration when using l as unit of length and t as unit of time, find its measure when using l_1 and t_1 respectively.

8. If 10 be the measure of an acceleration when using 5 feet and 4 seconds as units, find the measure if we use the units 4 feet and 5 seconds respectively.

9. If an acceleration when measured by the 'foot-second' system of units is 32, and is represented by 384 when a yard is unit of space, find unit of time.

10. How often does the measure of an acceleration of 120 yards per minute per minute contain the measure of an acceleration of an inch per second per second?

11. A point is moving with an acceleration of 10 (metre-seconds); find its measure in the (centimetre-second) units.

12. If 1760 yards per 60 seconds is the unit of velocity, and 3 feet be taken as the unit of length, what will be the unit of time?

13. If the unit of velocity be 32 feet per 16 seconds, and the unit of acceleration be 32 feet per second per second, find the unit of time.

14. If the unit of velocity be 12 feet per 3 seconds, and the unit of acceleration be 900 miles per hour per hour, find the unit of time.

15. If 11 yards per sec. per sec. be taken as the unit of acceleration, and 60 secs. be taken as the unit of time, required the unit of space.

16. If $\frac{2}{3}$ mile be the unit of space, and 15 minutes be unit of time, find (1) unit of velocity, (2) unit of acceleration, in the 'f.-s.' system.

17. If unit of velocity be 18 feet per $4\frac{1}{2}$ secs., and unit of acceleration be 2 fathoms per $\frac{1}{2}$ sec. per $\frac{1}{2}$ sec, find (1) unit of time, and (2) unit of space.

18. The measure of an acceleration is 32·12 in the foot-sec[illegible]nits; what is the unit of time in use when with a mile as unit of length the measure of the same acceleration is 87·6?

19. The measures of an acceleration and a velocity when referred to $(a+b)$ ft., $(m+n)$ secs. and $(a-b)$ ft., $(m-n)$ secs. respectively are in the inverse ratio of their measures when referred to $(a-b)$ ft., $(m-n)$ secs. and $(a+b)$ ft., $(m+n)$ secs.; their measures when referred to a ft., m secs. and b ft., n secs. are as $ma : nb$; show that $\frac{n^2}{m^2} = 1 - \frac{b^4}{a^4}$.

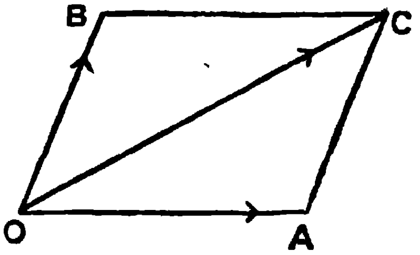
20. Two bodies P and Q are moving from rest with accelerations measured in the 'inch-second' and 'yard-minute' units respectively, and their measures are as 2 to 1. If P describe $(x+y)$ feet in $(a-b)$ seconds, and Q describe $(x-y)$ feet in $(a+b)$ seconds, and the velocities of P and Q measured in their respective units at the end of the $(a-b)$ and $(a+b)$ seconds are as 10 to 1, prove that $a:b=13:11$ and $x:y=43:7$.

58. Acceleration when uniform is measured by the increase of velocity per unit of time (Art. 51); hence any straight line which represents a given velocity in magnitude and direction will also represent in magnitude and direction the acceleration which is measured by that velocity added per unit of time.

PARALLELOGRAM OF ACCELERATIONS.

59. STATEMENT.—*If a point at any instant possesses two accelerations which are represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant acceleration is represented in magnitude and direction by the diagonal drawn through the point.*

We must show that the resultant acceleration of two accelerations which are represented in magnitude and direction by OA and OB will be represented in magnitude and direction by OC.



S[illegible] OA and OB represent
the accelerations,
∴ they represent the velocities added by each in unit of time.
The Parallelogram of Velocities (Art. 18)
is true for *all* velocities,
∴ it is true for velocity added in unit of time;
∴ OC will fully represent the resultant velocity
added in unit of time;
∴ OC will fully represent the resultant acceleration. (Q.E.D.)

60. The terms Component, Resultant, Resolution, Composition, apply to accelerations as well as to velocities, and we have analogous propositions. The *Parallelogram of Accelerations* has just been established, and in the same way we might prove the *Triangle of Accelerations*, the *Polygon of Accelerations*, etc.

UNIFORM ACCELERATION IN A STRAIGHT LINE.

61. *If a point moving with a uniform velocity* (u) *become subject to a uniform acceleration* (a) *in the direction of its motion, to find the velocity* (v) *after a given time* (t).

Our definition of the Measure of Uniform Acceleration (Art. 51) tells us that the velocity of the point is increased each second by the quantity a.

At the beginning the velocity $=u$.
At the end of 1^s ,, $=u+a$;
,, ,, 2^s ,, $=u+2a$;
,, ,, 3^s ,, $=u+3a$;
,, ,, t^s ,, $=u+ta$.
But the final velocity is denoted by v,

$$\therefore v=u+at.$$

EXAMPLES—XI.

1. A point moving with a velocity of 6 f.-s. has its velocity increased 8 feet per second per second; find its velocity after 12 seconds.

2. A stone thrown vertically downwards from the top of a cliff with a velocity of 40 f.-s. is subject to an acceleration of 32 f.-s.-s.; find its velocity after 3 seconds.

3. In what time will a point whose initial velocity is 10 acquire a velocity of 100 when moving with an acceleration of 6?

4. A point moving at the rate of 140 yards per minute has its velocity uniformly increased, so that at the end of a quarter of a minute it is moving at the rate of 15 miles an hour; find the measure of the acceleration in the foot-second system.

5. If a point moving with a uniform velocity of 100 f.-s. is subject to a constant *retardation* of 20 f.-s.-s., when will the particle be at rest?

62. *If a point moving with uniform velocity* (u) *become subject to a uniform acceleration* (α) *in the direction of its motion; to find the space* (s) *passed over in a given time* (t).

Since the velocity increases *uniformly* throughout the given time, the *mean* velocity during the interval will be *half the sum of the extreme velocities.*

To find the value of the Mean Velocity during t seconds, we have—

velocity at the beginning $= u$ (By data.)

" " end $= u + \alpha t$ (Art. 61.)

$\therefore$ sum of the velocities $= 2u + \alpha t$

$\therefore$ Mean Velocity $= u + \frac{\alpha t}{2}$.

Now the *space* described with uniform velocity in any time $=$ *velocity* × *time.* (Art. 9.)

$$\therefore s = \left(u + \frac{\alpha t}{2}\right)t.$$

$$\therefore s = ut + \tfrac{1}{2}\alpha t^2.$$

In Art. 76 another proof of this important result is given not involving the mean velocity.

EXAMPLES—XII.

1. A point has an initial velocity of 100 f.s., and moves with a uniform acceleration of 10 f.-s.-s.; find distance passed over in 5 seconds.

2. A point describes 124 feet in 4 seconds under a uniform acceleration of 8 f.-s.-s.; find the initial velocity.

3. A stone starting with an initial velocity of 29 f.-s. describes 750 feet with an acceleration of 32 f.-s.-s.; find the time.

4. If a point moving with an initial velocity of 12 f.-s. describes 252 feet under a uniform acceleration in 6 seconds, find the measure of the acceleration in the foot-second units.

63. *If a point moving with uniform velocity* (u) *describe a certain space* (s) *with a given acceleration* (α) *in the direction of its motion; to find the velocity* (v) *at the end of that space.*

We have $v=u+\alpha t$ (Art. 61); and $s=ut+\frac{1}{2}\alpha t^2$ (Art. 62);

$$\begin{aligned}\therefore\ v^2&=u^2+2u\alpha t+\alpha^2t^2\\&=u^2+2\alpha\left(ut+\tfrac{1}{2}\alpha t^2\right)\\&=u^2+2\alpha.s\\\therefore\ v^2&=u^2+2\alpha s.\end{aligned}$$

Then $v-u$ = velocity acquired while describing the distance s.

EXAMPLES—XIII.

1. A point has an initial velocity of 6 f.-s., and describes 8 feet with an acceleration of 4 f.-s.-s.; find its final velocity.

2. A point, having described 12 feet with an acceleration of 12 f.-s.-s, has a velocity of 18 f.-s.; find its initial velocity.

3. A point starting with initial velocity of 30 f.-s. describes 50 feet with an acceleration α, and has then a velocity of 40 f.-s.; find the value of α.

4. A point has a final velocity of 60 f.-s., having started with an initial velocity of 10 f.-s., and passed over 70 feet with uniform acceleration; find the measure of the acceleration in the foot-second units.

5. A point starting with an initial velocity of 12 f.-s. describes a certain space with an acceleration of 6 f.-s.-s., and has then a velocity of 30 f.-s.; find the space passed over.

64. Collecting the results obtained in Articles 61-63, we have—

(1) $v=u+\alpha t$ (connecting v and t),
(2) $s=ut+\frac{1}{2}\alpha t^2$ („ s „ t),
(3) $v^2=u^2+2\alpha s$ („ v „ s).

These may be called the '**Formulæ of Reference**' for uniformly accelerated motion in a straight line.

NOTE 1.—If a point have no initial velocity, it is said to *start from rest*. In this case we must assume $u=0$, in the 'Formulæ of Reference.'

NOTE 2.—If the acceleration be opposite to the direction of the initial velocity, then we must write a with a *negative* sign in the 'Formulæ of Reference.' (*See* Art. 44.)

NOTE 3.—A body falling freely under the earth's attraction has its velocity increased each second by 32·08 feet per second at the Equator, by 32·18 at Greenwich, by 32·25 at the Pole.

This case of uniform acceleration is so important that it is usual to denote its measure by the special letter 'g.'

If nothing be said to the contrary, the student may assume that $g=32$.

Example i.—A point moving at the rate of 44 feet per second is brought to rest in 30 seconds by a constant retardation; find the measure of the retardation.

Here $u=44$, $v=0$, $t=30$, and a is required.

We evidently make use of $v=u+at$;

$$\therefore 0=44-a.30;$$

$$\therefore a=1\tfrac{7}{15} \text{ feet per sec. per sec.}$$

Hence we infer that the velocity is *diminished* at the rate of $1\frac{7}{15}$ feet per second per second; and thus in 30 seconds the point has *lost all* its velocity.

Example ii.—A particle is thrown up with a velocity of 144 f.-s.; after what time will its height be 320 feet?

If we agree to consider velocity, acceleration, and space, measured in one direction as positive, then when measured in the opposite direction they must be regarded as negative. In this question the initial velocity and the height are measured *up*, and the acceleration caused by gravity is measured *down*. If, therefore, we consider quantities measured up as positive, then quantities measured down will be negative.

Here $u=144$; $s=320$; $a=g=-32$; and t is required.

We evidently make use of $s=ut+\frac{1}{2}at^2$;

$$\therefore 320=144t-\tfrac{1}{2}\times 32\times t^2;$$

$$\therefore 16t^2-144t+320=0;$$

$$\text{whence } t=4, \text{ or } 5.$$

We conclude therefore that the particle is at the height of 320 feet in 4 seconds and 5 seconds after its projection; in 4 seconds *on its way up*, in 5 seconds *on its way down.*

Example iii.—A particle is projected vertically upwards with a velocity of 160 f.-s.; find (*a*) how high it will rise.

(*b*) the time of ascent;

(*c*) the velocity at the height of 300 feet;

(*d*) the velocity after $3\frac{1}{2}$ seconds;
(*e*) the velocity after 6 seconds.

Here $u=160$; $a=g=-32$.

(a) *Height attained*—

$$v^2=u^2+2as;$$
$$\therefore 0=160^2-2\times 32\times s;$$
$$\therefore s=400 \text{ feet.}$$

(b) *Time of ascent*—

$$v=u+at;$$
$$\therefore 0=160-32t;$$
$$\therefore t=5 \text{ seconds.}$$

(c) *Velocity after describing* 300 *feet*—

$$\begin{aligned} v^2&=u^2+2as \\ &=160^2-2\times 32\times 300 \\ &=25{,}600-19{,}200 \\ &=6{,}400 \end{aligned}$$
$$\therefore v=80 \text{ feet per second.}$$

(d) *Velocity after* $3\frac{1}{2}$ *seconds*—

$$\begin{aligned} v&=u+at \\ &=160-32\times\frac{7}{2} \\ &=160-112 \\ &=48 \text{ feet per second.} \end{aligned}$$

i.e. a velocity of 48 ft. per sec. *upwards.*

(e) *Velocity after* 6 *seconds*—

$$\begin{aligned} v&=u+at \\ &=160-32\times 6 \\ &=160-192 \\ &=-32 \text{ feet per second;} \end{aligned}$$

i.e. a velocity of 32 feet per second *downwards.*

Example iv.—A particle passes a point 528 feet high with a velocity of 128 f.-s. up; after what time will it return to the ground?

Let us consider the positive direction *down*, and the negative direction *up.* The ground is therefore 528 feet *below* the point.

Here $s=528$; $a=32$; $u=-128$.

We use the formula $s=ut+\frac{1}{2}at^2$;

$$\therefore 528=-128t+16t^2;$$
$$\therefore t^2-8t-33=0;$$
$$\text{whence } t=11 \text{ seconds.}$$

The other root, -3, of the equation, signifies that the particle will return to the ground in 3 seconds after passing the point *on its way down.*

Note.—It is much simpler to work this question as shown than to find (1) the further height to which the particle will ascend; (2) the time of this further ascent; (3) the time of descending through [illegible] whole space ascended from the ground; (4) and, finally, by (2), (3) the whole time required.

Example v.—A particle is dropped from a height of $12g$, and 4 seconds afterwards another is projected from the ground vertically towards it with a velocity of $4g$; *when* and *where* will they meet?

When?—Let t seconds elapse after 2nd is projected.

Then $(t+4)$ „ 1st was let fall.

Space described by 1st, $s_1 = 0 + \frac{1}{2} \times g \times (t+4)^2$, because $u = 0$;

„ 2d, $s_2 = 4g \times t - \frac{1}{2} \times g \times t^2$; (*See* **Ex.** ii.)

$$\therefore s_1 + s_2 = \frac{g}{2}(t+4)^2 + 4gt - \frac{g}{2}.t^2.$$

$$\text{But } s_1 + s_2 = 12g \quad \text{(by data)};$$

$$\therefore 12g = \frac{g}{2}(t+4)^2 + 4gt - \frac{g}{2}t^2;$$

$$\therefore 24 = (t+4)^2 + 8t - t^2;$$

$$24 = t^2 + 8t + 16 + 8t - t^2;$$

$$\therefore 16t = 8;$$

$$\therefore t = \tfrac{1}{2} \text{ second.}$$

Where?—Let x be the height above the ground.

$$\text{Then } x = 4g \times \tfrac{1}{2} - \tfrac{1}{2}g\left(\tfrac{1}{2}\right)^2$$
$$= 64 - 4$$
$$= 60 \text{ feet.}$$

The particles therefore meet ½ sec. after the second is projected, and at a height of 60 feet.

Examples—XIV.

1. A body falls for 5 seconds; find its velocity and the space described.

2. A particle is projected vertically upwards with a velocity of 250 f.-s.; how high will it rise?

3. A particle is projected vertically upwards, and has lost half its original velocity when it has risen through 60 feet; find the initial velocity.

4. A particle is projected vertically upwards with a velocity of 320 f.-s.; find the velocity after it has described 576 feet.

5. A particle has an initial velocity of 12 f.-s., and having described

30 feet has a velocity of 18 f.-s.; find the acceleration, and also the time of motion.

6. A particle is thrown upwards with a velocity of 200 f.-s.; after what interval will it be 50 feet above the ground?

7. A particle is projected from *A* with an initial velocity of 100 f.-s., and is subject to a retardation of 40 f.-s.-s.; find the distance from *A* after 5 seconds.

8. A particle under uniform acceleration described 180 feet in 6 seconds from rest; find the acceleration.

9. A particle is projected vertically upwards with a velocity of 640 f.-s.; in what time will it be at a height of 624 feet? Explain the double result.

10. Two particles fall from heights of 256 and 1600 feet respectively; if they reach the ground at the same instant, find the interval between their starting.

11. A particle starts from rest under a uniform acceleration of 16 f.-s.-s.; after 5 seconds another particle is projected from the same point, after it and with the same acceleration. If the second body overtake the other in 10 seconds, find the velocity of projection.

12. A body is dropped from a height of 784 feet; after it has fallen through 144 feet, a second body is projected vertically downwards from the same point with such a velocity that the bodies reach the ground in the same time; find the velocity of projection.

13. A body is dropped from a height of 576 feet; after it has fallen through 48 feet, a second body is projected vertically downwards, and the bodies reach the ground at the same instant; find the velocity of projection.

14. A particle projected vertically up passes a point 176 feet high with a velocity of 160 f.-s.; after what interval will it return to the ground?

15. A particle thrown up passes a point 240 feet high with a velocity of 240 f.-s. up; when will it return to the ground?

16. A particle thrown up passes a point *a* feet high with a velocity of *b* f.-s.; after what time will it return to the ground?

17. A ball is let fall from a height of 256 feet, and at the same instant a second ball is projected vertically upwards to meet it; find the velocity of projection of the latter if the balls meet at a height of 64 feet.

18. A ball is projected vertically upwards with a velocity of $5g$; after what time must another be projected with a velocity of $7g$ in order that

the latter going up may meet the other coming down at a height of 384 feet?

19. A body projected vertically up is at the same height at the end of 5 and 9 seconds; find the initial velocity.

20. A body is thrown up with an initial velocity of 176 f.-s.; after 1½ seconds another is projected up, and overtakes the first at its highest point; find the velocity with which the latter is projected.

21. A body is let fall from a given point; 3 seconds later another is let fall from a point 240 feet below; in what time will the bodies pass each other, and how far apart will they be one second after that happens?

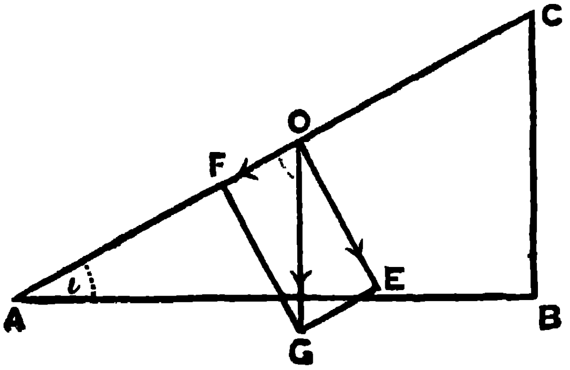
22. From the top of a tower 100 feet high a stone is dropped; with what velocity must another be projected downwards from the same point after the lapse of half a second, that the stones may reach the ground at the same instant?

23. One second after the first stone in Ex. 22 was let fall, another was projected downwards so as to reach the ground half a second before the other; find (1) the velocity of projection; (2) where the stones passed each other.

24. A body shot vertically upwards rises 240 feet in 4 seconds; how high will it rise, and what time will elapse between its starting and returning to the point of projection?

65. We next proceed to *find the acceleration with which a particle will slide down an inclined plane.*[1]

Let AC be a plane inclined to the horizon at an angle $CAB=i$. Draw a vertical line OG, and let it represent the value of g. Resolve the acceleration g along the plane and at right angles to it. Let these component accelerations be OF and OE respectively. Then (Art. 28) the component



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[1] The assumptions made in this Article will be justified in Article 133 after the student has read the Laws of Motion.

OF will be the *resolved part* along the plane of the acceleration *OG*.

Therefore the *total* effect of *g* along the plane will be fully represented by *OF*.

$$\text{Now } OF = OG \cos FOG;$$
$$\therefore\ OF = OG \sin BAC;$$
$$\text{or, } \alpha = g \sin i.$$

And when motion takes place on an inclined plane whose inclination is *i*, we must substitute '$g \sin i$' for 'α' in the 'Formulæ of Reference.' (Articles 61-63.)

EXAMPLES—XV.

1. Find the angle of inclination, when the acceleration along the plane is $\frac{1}{2}g$.

2. In what time would a body slide down a plane 640 feet long, having an inclination to the horizon = 30°?

3. A particle slides from rest for $4\frac{1}{2}$ feet down an inclined plane, and has a velocity of 12 f.-s. at the bottom; find the inclination.

4. A particle takes twice as long to slide down the length as to fall through the vertical height of a given plane; find the inclination of the plane.

5. In what time will a particle slide down an inclined plane whose length is 100 feet, and height 65 feet?

6. A particle slides down a plane 150 feet long in 12 seconds; find the height of the plane.

7. Find the velocity acquired by a particle on sliding down a smooth plane whose base is 200 feet and inclination 45°.

8. If the time of sliding down an inclined plane : the time of falling down the vertical height $= 2\sqrt{2} : \sqrt{3} - 1$; find the inclination.

9. A plane rises 1 foot in every 100 feet of length; find the velocity with which a particle must be projected up the plane in order that it may describe 1839 feet in 10 seconds.

10. A particle slides down a plane 100 feet long, inclined to the horizon at 60°; find the time of describing the upper and lower halves respectively.

11. A particle slides from rest down a plane 320 feet long, and having an inclination of 30°; find the time of describing the last 50 feet.

12. A particle takes twice as long to slide down either of the sides of an isosceles triangle as to fall freely from the vertex to the base (which is horizontal); find the vertical angle of the triangle.

13. With what velocity must a particle be projected down a plane of given length (l) whose inclination is 30°, so as to reach the bottom simultaneously with another particle which starts at the same instant and falls freely through the vertical height of the plane?

14. Show how to place a smooth plane of given length (l) so that a particle may acquire a given velocity (v) in sliding down it.

66. *A point is moving with an initial velocity* (u), *and becomes subject to a uniform acceleration* (α) *in the direction of motion; to find the space described in any particular* (n^{th}) *second.*

Space in n^{th} second = space in n seconds − space in $(n-1)$ seconds.

Space in n seconds $= u.n + \frac{1}{2}\alpha.n^2$. (Art. 62.)

" $(n-1)$ " $= u.(n-1) + \frac{1}{2}\alpha.(n-1)^2$.

$\therefore$ Space in n^{th} second $= u(n-n+1) + \frac{1}{2}\alpha\,(n^2-n^2+2n-1)$

$= u + \frac{1}{2}\alpha\,(2n-1)$.

If the point have no initial velocity, then $u=0$, in this result.

NOTE.—The student may, as an exercise, obtain this result by using the *mean velocity* in n^{th} second.

Example i.—Find the space described in the 6th second by a particle falling from rest.

If S_n denote the space described in n seconds,

Then $s_6 = 0 + \frac{1}{2}g\,(6)^2$,

And $s_5 = 0 + \frac{1}{2}g\,(5)^2$;

$\therefore$ Space in 6th second $= \frac{1}{2}g\,(11) = 16 \times 11 = 176$ feet.

Example ii.—A particle has fallen from rest for 16 seconds; compare the spaces described in the 6th and 16th seconds.

$$s_6 - s_5 = \tfrac{1}{2}g(6)^2 - \tfrac{1}{2}g(5)^2 = \tfrac{1}{2}g \times 11.$$

$$s_{16} - s_{15} = \tfrac{1}{2}g(16)^2 - \tfrac{1}{2}g(15)^2 = \tfrac{1}{2}g \times 31;$$

$\therefore$ Space in 6th sec. : space in 16th sec. $= 11 : 31$.

Example iii.—A particle which had an initial velocity of 10 f.-s. described 54 feet in the 6th second; find the acceleration.

$$s_6 = 10.6 + \tfrac{1}{2}a\,(6)^2,$$
$$s_5 = 10.5 + \tfrac{1}{2}a\,(5)^2;$$
$$\therefore\ 54 = 10 + \tfrac{1}{2}a\,(11); \qquad \therefore\ a = 8 \text{ f.-s.-s.}$$

Example iv.—A particle moving with uniform acceleration described 140 feet between the 3d and 5th seconds, and 770 feet between the 12th and 17th seconds; find the initial velocity and the acceleration.

Let u denote the initial velocity, and let a denote the acceleration.

Then $s_5 = u.5 + \tfrac{1}{2}a\,(5)^2,$

And $s_3 = u.3 + \tfrac{1}{2}a\,(3)^2;$

$$\therefore\ 140 = 2u + \tfrac{1}{2}a.16 \quad \ldots\ldots \quad (1.)$$

Also $s_{17} = u.17 + \tfrac{1}{2}a\,(17)^2,$

And $s_{12} = u.12 + \tfrac{1}{2}a\,(12)^2;$

$$\therefore\ 770 = 5u + \tfrac{1}{2}a.145 \quad \ldots\ldots \quad (2.)$$

From these two equations (1) and (2) we get $u = 38$ f.-s. and $a = 8$ f.-s.-s.

EXAMPLES—XVI.

1. A stone has fallen for 7 seconds; find the space passed over in the seventh second.

2. A point moving from rest with uniform acceleration describes 590 feet in the 30th second; find the acceleration.

3. A point had an initial velocity of 100 f.-s., and it described 328 feet in the 10th second; find the acceleration.

4. A body has fallen for 12 seconds; compare the spaces passed over in the 6th and last seconds.

5. A body moving from rest with uniform acceleration describes 300 feet in the 5th second of its motion; find the space passed over in the 8th second, and in 8 seconds

6. A body whose velocity was uniformly accelerated described 84 feet and 108 feet in the n^{th} and $(n+1)^{th}$ sec. respectively from rest; find the acceleration.

7. A body whose velocity was uniformly accelerated described 82 feet in the 7th second, and 98 feet in the 9th second of its motion; find both the initial velocity and the acceleration.

8. A falling body describes 16/25 of the whole space in the last second; find the whole time of motion.

9. A stone falls over the edge of a cliff, and describes 7/16 of the height of the cliff in the last second of its fall; find the height of the cliff.

10. A stone falls from an anchored balloon, and describes 8/9 of the height in the last second of its fall; find the height of the balloon.

11. The space described by a falling body in the 5th second : space described in the last $=1:3$; find the height from which the body is let fall.

12. The space described by a falling body in the last second of its fall : space in the last second but one $=15:13$; find the height from which the body is let fall.

13. A falling body describes in the last second a space equal to three times that described in the last second but five; how long has it been falling?

14. A particle sliding from rest down an inclined plane whose angle of inclination is $\tan^{-1}(3/4)$, describes 86·4 in the last second but one; find the length of the plane.

15. A point moves with uniform acceleration, and in the m^{th}, n^{th}, p^{th} seconds it describes x, y, z feet respectively; show that

$$x(n-p)+y(p-m)+z(m-n)=0.$$

67. We can obtain the following interesting results from the 'Formulæ of Reference.' Examples in which some of them are required will be found in the Exercises at the end of this Chapter.

68. *When a point starts with no initial velocity, and moves with uniform acceleration, the space described varies as the square of the time.*

We have $s=ut+\frac{1}{2}at^2$. (Art. 62.)

And, by data, $u=0$;

$\therefore s=\frac{1}{2}at^2$;

but $\frac{1}{2}a$ is a constant;

$\therefore s \propto t^2$. (Q.E.D.)

69. *When a point moves with uniform acceleration the spaces described in successive seconds are in Arithmetical Progression.*

The space in the n^{th} second $=u+\frac{1}{2}a(2n-1)$; (Art. 66.)

$\therefore$ space in 1st sec. $=u+\frac{1}{2}a(2-1)=u+\frac{1}{2}a$;
" 2nd " $=u+\frac{1}{2}a(4-1)=u+\frac{3}{2}a$;
" 3rd " $=u+\frac{1}{2}a(6-1)=u+\frac{5}{2}a$.

Hence the spaces described in successive seconds form an Arithmetical Progression, whose common difference is a.

The student will notice that these spaces are proportional to the *odd* numbers 1, 3, 5, . . . when the body starts from rest.

NOTE.—The converse of this theorem is given as an exercise in the Examples at the end of this Chapter. *See* Example 111.

70. *The space described by a particle falling freely for a given time is approximately equal to the square of the number of quarter-seconds.*

We know that $s=ut+\frac{1}{2}at^2$; and in this special case $u=0$, and $g=32$, nearly;

$$\therefore s=16t^2;$$
$$=(4t)^2;$$
$$=(\text{number of quarter-seconds})^2. \quad (\text{Q.E.D.})$$

71. *When a particle is thrown vertically upwards its time of ascent is equal to its time of descent.*

Let t=time of ascent, and T=time of ascent and descent.

We know that $v=u+at$.

In this formula $v=0$, because after t seconds the body is at its highest point; and $a=-g$, because its direction is opposite to the direction of u.

$$\therefore 0=u-gt; \qquad \therefore t=\frac{u}{g}.$$

When the body has returned to the point of projection, its height above that point $=0$, and this is the space described in the time $=T$.

We know that $s=ut+\frac{1}{2}at^2$;

$$\therefore \quad 0=uT-\frac{1}{2}gT^2;$$

$$\therefore gT^2 = 2uT;$$

$$\therefore \quad T = \frac{2u}{g} = \text{time up and down},$$

$$\text{and } t = \frac{u}{g} = \text{time up};$$

$$\therefore T - t = \frac{u}{g} = \text{time down}.$$

Therefore, the time of ascent is equal to the time of descent. (Q.E.D.)

The time up and down—that is, the *time in the air*—is called the *Time of Flight.*

72. *When a particle is thrown vertically upwards, on returning to the point of projection it has a velocity equal to the velocity of projection, but in the opposite direction.*

Let u = velocity of projection,
and v = velocity on returning to the point of projection.

$$\text{Then, Time of Flight} = \frac{2u}{g}. \qquad \text{(Art. 71.)}$$

$$\text{Now } v = u + at;$$

$$\therefore \quad v = u - g \cdot \frac{2u}{g};$$

$$= u - 2u;$$

$$= -u.$$

Or, we may establish the theorem in the following very simple manner.

When the particle returns to the ground its height above the ground = 0.

To find the velocity at this height we use the formula $v^2 = u^2 + 2as$.

$$\text{But } s = 0;$$

$$\therefore v^2 = u^2;$$

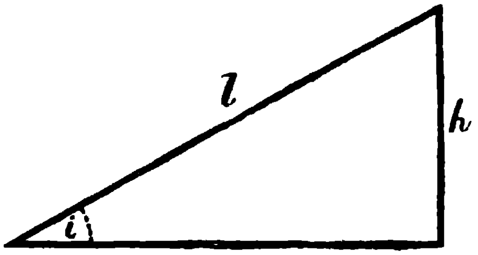
$$\therefore v = \pm u,$$

i.e. $+u$ is the velocity of projection, and $-u$ is the velocity on striking.

Therefore the velocity on returning to the point of projection is equal in magnitude to the velocity of projection, but in the opposite direction. (Q.E.D.)

COR.—Hence the velocity at *any* point in the ascent is equal and opposite to the velocity at the same point in the descent. The student will notice that this is a special case of a general theorem, viz. :—'If a particle is moving in a straight line, and its velocity be uniformly retarded, it will pass twice through any point in its path with its velocities equal in magnitude but opposite in direction.'

73. *When a particle slides from rest down an inclined plane, the velocity acquired in describing the length of the plane is equal to the velocity which it would acquire in falling freely through the height of the plane.*



Let v_1 = velocity acquired in describing the length of the plane.

We use the formula

$$v^2=u^2+2as.$$

In this, $s=l$; $u=0$; $a=g\sin i$,

$$=g\times\frac{h}{l};$$

$$\therefore v_1^2=0+2g\times\frac{h}{l}\times l$$

$$=2gh;$$

$$\therefore v_1=\sqrt{2gh}\;.\;\;.\;\;.\;\;.\;\;.\quad(1.)$$

Let v_2 = velocity acquired in falling through the height.

$$\therefore v_2^2=0+2gh;$$

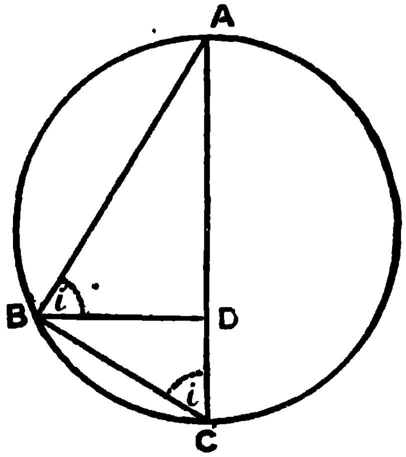
$$\therefore v_2=\sqrt{2gh}\;.\;\;.\;\;.\;\;.\;\;.\quad(2.)$$

Hence, from the results (1) and (2), we conclude that

$$v_1=v_2.\quad\text{(Q.E.D.)}$$

74. *The time occupied by a particle falling from rest down any chord of a vertical circle, passing through its highest point, is equal to the time taken to fall down the vertical diameter.*

To show that the time down any such chord AB = time down AC.



Draw $BD \perp AC$, and join BC.

Now $\angle ABD = \angle ACB$; $\therefore$ each $= 90° - CBD$;

$$\therefore \sin ABD = \frac{AB}{AC}.$$

We use the formula $s = ut + \frac{1}{2}at^2$.

Let the time down $AB = t_1$,

$$\therefore AB = 0 + \tfrac{1}{2}g \sin i \times t_1^2;$$

$$\therefore t_1^2 = \frac{2AB}{g \sin i} = \frac{2AB}{g \times \frac{AB}{AC}} = \frac{2AC}{g} \quad \ldots\ldots \quad (1.)$$

Let the time down $AC = t_2$.

$$\therefore AC = 0 + \tfrac{1}{2}gt_2^2;$$

$$\therefore t_2^2 = \frac{2AC}{g} \quad \ldots\ldots \quad (2.)$$

Hence, from the results (1) and (2), we conclude that

$t_1 = t_2$. (Q.E.D.)

75. In the same way it may be shown that the time down any chord of a vertical circle passing through its *lowest* point is equal to the time down the vertical diameter.

COR.—If from a given point particles at any instant fall away in all directions, at the end of a time t they will all be situated on the surface of a sphere whose diameter is $\frac{1}{2}gt^2$.

EXAMPLES—XVII.

1. If two vertical circles touch at their highest points, and a straight line be drawn through this point cutting the circles, show that the time down the part between the circumferences is constant.

2. A particle moves down a chord of a vertical circle inclined at an angle of 60° to the horizon, and passing through the lowest point of the circle. If it reach that point with a velocity equal to that due to the vertical diameter, prove that its velocity is doubled during its descent.

3. Determine that diameter of a vertical circle, down the latter half of which a body (having started from rest at the other extremity) falls in the same time as down the whole vertical diameter.

4. If a circle be placed in a vertical plane, determine that chord passing through its lowest point down which a body must fall so that it may acquire the greatest horizontal velocity.

5. Find that radius of a vertical circle so that the time to the centre may be equal to the time from the extremity of the radius to the lowest point of the circle.

6. AB is the horizontal diameter of a vertical circle, C the lowest point of the circle, P a point on the circumference. If t_1, t_2, t_3, the times of falling from rest down PA, PB, PC, be such that $t_1^2+t_2^2=4t_3^2$, find the angle PAB.

7. If the angle PAB in the last question be 15°, show that $t_1-t_2=t_3\sqrt{2}$.

8. ABC is a vertical circle, A its highest point, CB a diameter which is produced to meet the tangent at A in D; if a particle fall down DC from rest, find (1) its velocity at B; (2) its velocity at C; (3) the angle ADC, so that the velocity at B may be half the velocity acquired in moving from B to C.

9. If AB be the horizontal diameter of a vertical circle, show that the time down any chord AP varies inversely as the time down the chord BP.

76. Although the proof of the Proposition in Art. 62 is sufficient, yet some may desire to have it established without reference to the mean velocity.

‘*A point has initial velocity* (u), *and an acceleration* (a) *in the direction of motion is imparted to it; it is required to show that in any time* (t) *the space described* $=ut+\frac{1}{2}at^2$.’

Let the time t be divided into n equal intervals,

$$\therefore \text{ each interval} = \frac{t}{n}.$$

The velocity added in the interval $\frac{t}{n} = \frac{at}{n}$.

The number of intervals will be

$$1, \quad 2, \quad 3, \quad . \quad . \quad . \quad . \quad . \quad n.$$

The velocities at the *beginning* of these intervals will be

$$u, \quad u+\frac{at}{n}, \quad u+\frac{2at}{n}, \quad . \quad . \quad . \quad . \quad u+\frac{(n-1)at}{n}.$$

The velocities at the *end* of these intervals will be

$$u+\frac{at}{n}, \quad u+\frac{2at}{n}, \quad u+\frac{3at}{n}, \quad . \quad . \quad . \quad . \quad u+\frac{nat}{n}.$$

Let s_1 and s_2 denote the whole space described, if the point moved uniformly during each interval with the velocity which it possessed at the *beginning* and *end* of the intervals, respectively; then, since $s=vt$ (Art. 9), we have

$$s_1 = u\cdot\frac{t}{n} + \left(u+\frac{at}{n}\right)\frac{t}{n} + \left(u+\frac{2at}{n}\right)\frac{t}{n} + \ . \ . \ . \ + \left(u+\frac{\overline{n-1}.at}{n}\right)\frac{t}{n};$$

$$= n\cdot\frac{ut}{n} + \frac{at^2}{n^2}(1+2+3+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \overline{n-1} \text{ terms});$$

$$= ut + \frac{at^2}{n^2}\cdot\frac{n}{2}(n-1), \text{ by Arith. Prog.};$$

$$= ut + \frac{at^2}{2}\left(1-\frac{1}{n}\right).$$

In the same way the student may show that

$$s_2 = ut + \frac{at^2}{2}\left(1+\frac{1}{n}\right).$$

Now the actual space (s) described must lie between these values of s_1 and s_2 whatever be the value of n.

When n is increased the quantity $\frac{1}{n}$ is diminished, and in the limit vanishes.

Hence when n is endlessly increased, s_1 and s_2 will each become ultimately equal to $ut+\frac{at^2}{2}$, which therefore represents the actual space described. $\therefore s = ut + \frac{1}{2}at^2$. (Q.E.D.)

LINES OF QUICKEST DESCENT.

77. By means of the two results arrived at in Arts. 74 and 75, we are enabled to solve a series of problems known as 'Lines of Quickest Descent.'

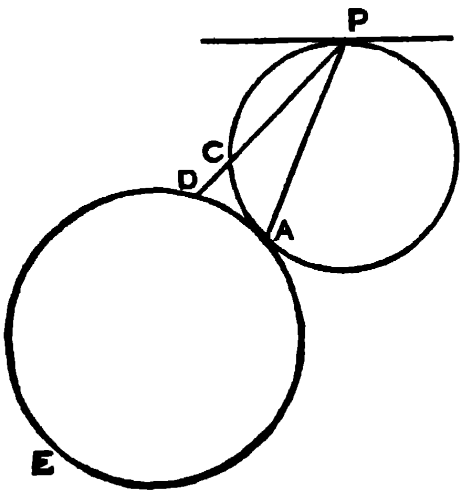
78. Def.—**A Line of Quickest Descent** is the line along which a particle will pass in the shortest time from one assigned position to another; *e.g.* from a given point to a given line, or to a given circle, or the converse.

79. The student is advised to become familiar with the solution of the following problems in Geometry before attempting to find any line of quickest descent.

1. *Draw a circle which shall touch two given straight lines, touching one of them at a given point.*
2. *Draw a circle which shall touch a given straight line at a given point, and also touch a given circle.*

   Take various cases, *e.g.* when the point is first within, and then without the circle.
3. *Draw a circle which shall touch a given circle at a given point and also touch a given straight line.*

Example i.—Find the line of quickest descent from a given point without a circle to the circle.



Let P be the point; ADE the circle. Draw a horizontal line through P. Then describe a circle which shall touch this line in P, and also touch the circle ADE (No. 2 above). Let the circles touch at A, and draw PA and any line PCD cutting the circles at C and D respectively.

PA is the line required.

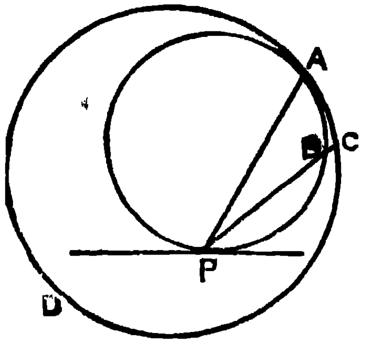
The time down PA = time down PC (Art. 74),

and $\therefore <$ time down PD.

Similarly for all other lines through P which cut the circles.

$\therefore$ PA is the line of quickest descent from P to the circle ADE.

Example ii.—Find the line of quickest descent from a circle to a given point within it.



Through the given point P draw a horizontal line. Describe a circle which shall touch this line at P and also touch the circle ACD. Let the circles touch at A; then AP is the line of quickest descent from the circle to P. Draw *any* straight line CBP cutting the circles and passing through P.

Then time down AP = time down BP (Art. 75),

and $\therefore <$ time down CP.

EXAMPLES—XVIII.

NOTE.—The circles, lines, and points are supposed to be in the same vertical plane.

1. Find the line of quickest descent—
   (*a*) From a given point to a given straight line.
   (*b*) From a given straight line to a given point.
   (*c*) From a given straight line without a given circle to the circle.
   (*d*) From a given circle to a given point without it.
   (*e*) From a given circle to a straight line without it.
   (*f*) From a given point within a circle to the circle.
   (*g*) From a given circle to a given point within it.
   (*h*) From a given circle to another circle within it.
   (*i*) From a given circle within the circle to the circle.
   (*j*) From a given circle to another circle without it.

2. A point P is 16 feet from a plane inclined at an angle of 30° to the horizon and above the plane. Find the *time* of quickest descent in a straight line from P to the plane.

3. C is the centre of a vertical circle; B any point within it vertically above C. If BD be drawn horizontally to meet the circumference in D, and CE be taken in CD, $= CB$, show that the line of quickest descent from E to the circumference is perpendicular to CD.

MISCELLANEOUS EXERCISES—XIX.

1. Compare the velocities of two particles, one of which moves through 13 yds. in $2\frac{1}{4}$ secs., and the other through 260 ft. in a minute.

2. Compare the velocities of two particles, one of which moves at the rate of 8 miles an hour, the other at the rate of 176 feet per minute.

3. What is implied when we say that the acceleration due to gravity is 32·2?

4. A train increases its speed uniformly from 30 to 40 miles per hour in 12 minutes; compare its acceleration with that of a particle falling freely.

5. A man walks with a velocity represented by 5·5, and finds that he travels 10 miles in 3 hours; if 2 secs. be the unit of time, find the unit of space.

6. Distinguish between *uniform velocity* and *uniform acceleration;* and give an instance of each.

7. State and prove the theorem known as the 'Parallelogram of Velocities.'

8. Explain and give illustrations of the following varieties of motion:— *translation, rotation, undulation.*

9. What is the acceleration (f.-s.), supposed uniform, of a train which while passing over 120 yards changes its velocity from 15 miles an hour to 60 miles an hour?

10. A point describes 45 feet and 55 feet in two successive seconds; find the space it would describe (1) in 20 seconds from rest, (2) in the 20th second from rest.

11. Establish the formula, $s=ut+\frac{1}{2}at^2$.

12. Assuming the formulæ, $v=u+at$, and $s=ut+\frac{1}{2}at^2$, establish the formula $v^2=u^2+2as$.

13. A particle projected vertically upwards passes a point 640 feet high with a velocity of 96 f.-s.; after what time will it return to the ground?

14. A particle thrown up is 10 seconds in the air; find (1) its velocity of projection; (2) the height to which it rises; (3) the time taken to rise 200 feet; (4) its velocity at a height of 300 feet.

15. A particle is projected vertically upwards with a velocity which would carry it to a height of 576 feet; after what time will it be coming down with a velocity of 48 f.-s.?

16. Show that the unit of acceleration varies directly as the unit of length, and inversely as the square of the unit of time.

17. Show that the velocity acquired by a body falling down an inclined plane is the same as if it had fallen through the vertical height of the plane.

18. A point moves with a velocity increasing every minute at the rate of 50 yards per minute. What space is passed over during the interval between its having a velocity of 20 and one of 80 yards per minute?

19. If 3 seconds were substituted for 1 second as the unit of time, by what number would an acceleration one-fourth as great as that caused by gravity be represented?

20. A train starts from A; its pace is quickened so that in 2 minutes its velocity is 30 miles an hour; it then travels for 18 minutes with uniform velocity; its speed then is regularly diminished so that in two minutes the train comes to rest at B. Find the distance between A and B.

21. A stone let fall from a certain height would reach the ground in 5 seconds. If it were stopped at the end of 3 secs. and afterwards let fall, how long would it take to fall through the rest of the distance?

22. A body falls from a certain point; after it has been in motion for 3 secs. it moves for 2 secs. more with the uniform velocity which it has acquired; find the whole space described.

23. Show that the time down *any* chord of a vertical circle passing through its lowest point is equal to the time down the vertical diameter.

24. A man walks with a velocity represented by $2\frac{1}{3}$, and he finds that he walks $7\frac{1}{2}$ miles in $2\frac{1}{4}$ hours; if 3 secs. be the unit of time, what is the unit of length?

25. A body projected vertically upwards loses 2/3 of its velocity in describing 9 feet. To what height will it rise?

26. A particle describes spaces of 24 feet and 64 feet respectively in two successive intervals each of 4 seconds duration; required the measure of the acceleration.

27. In 3 minutes after starting from a station a train is travelling at the rate of 40 miles an hour; express its mean acceleration in the interval with a foot and a second as units.

28. Find the vertical and horizontal components of the acceleration down a plane whose inclination is i.

29. Define *velocity*, and explain how it is measured (1) if uniform, (2) if variable.

30. If gravity be represented by 14 instead of 32, when the unit of time is 5 seconds, find the unit of length.

31. If 3 seconds be the unit of time, and a square yard be represented by 16, find the measure of a velocity of 15 miles an hour.

32. In what time will a body slide down an inclined plane whose length is 100 feet, and height 64 feet? What is the velocity acquired?

33. A body sliding down an inclined plane describes 104 feet in the 7th second from rest; find the angle of inclination.

34. In the last question, where must the plane be divided in order that the times of describing the two portions may be equal?

35. Explain how the measures of a *velocity* and an *acceleration* respectively depend on the units of length and time.

36. A particle moves from rest under a uniform acceleration, and describes 20 yards in a certain second, and 30 yards in the next second. Compare this acceleration with that caused by gravity, and find how far the particle had moved from rest before the first observation was made.

37. If each unit employed in the measurement of an acceleration be increased x times, show that the new value of the measure of the acceleration will be x times its former value.

38. What is the unit of length when a minute is the unit of time, and a velocity of 1 f.-s. is denoted by the cube root of the number which denotes an acceleration of 60 f.-s.-s.?

39. A man runs with a velocity 5 on a horizontal platform which is rising vertically with a velocity 12; find the magnitude and direction of his actual motion.

40. Prove that the velocity of a particle moving from rest under a constant acceleration varies directly as the square root of the distance described.

41. Obtain the formula $v^2 = u^2 + 2as$, defining the symbols employed.

42. A particle subject to a constant acceleration describes 50 and 56 feet in the 6th and 7th secs. respectively; find the initial velocity.

43. If at the two instants when it is 1 foot distant from a certain point in its path a particle is moving with velocities of 1 f.-s. and 7 f.-s. respectively, find (1) the value of the acceleration; (2) the velocity with which it passes through the point.

44. A person at a window throws a ball up outside, and observes that in 3 seconds it passes him on its downward path, and that in another second it strikes the ground. Find (1) the height of the window above the ground; (2) the vertical velocity of projection.

45. A stone falls freely for 3 seconds when it breaks a pane of glass, and thereby loses one-half its velocity; find the height of the glass above the ground if the stone reaches the ground in 2 seconds after breaking the glass.

46. If a stone fall freely for 4 seconds, and then perforating a roof lose one-third of its velocity, find the entire space described in $5\frac{1}{2}$ seconds.

47. A particle moves over 16, 40, 70 feet in the 1st, 2nd, 4th seconds respectively. Is it moving with uniform acceleration?

48. A particle falls down a smooth inclined plane; if v_1, v_2 are its velocities at two points whose vertical distance apart is h, prove that $v_1^2 \backsim v_2^2 = 2gh$.

49. A particle moving with uniform acceleration is observed to pass over a feet and b feet respectively in two consecutive seconds; find the acceleration, and find the time from rest before describing a.

50. A stone in falling from rest describes the n^{th} part of the height in the last second; find the time of falling.

51. What is the acceleration (a yard and a second being the units) of a train which quickens uniformly in one minute from 10 miles to 60 miles an hour.

52. If a body falling freely for 1 second acquire 5 units of velocity, and its acceleration be 10, find the units of length and time.

53. If the unit of velocity be a velocity of a mile an hour, and the acceleration due to gravity be represented by $327\frac{3}{11}$, what are the units of length and time? Express the unit of acceleration in the foot-second units.

54. If the space described in the last 5 seconds of motion under uniform acceleration (a) be 11 times that described in the first five seconds, find the whole time of motion, the initial velocity being $6a$.

55. Can a particle move with a velocity of constant magnitude, and yet be moving with an acceleration?

56. A stone is projected vertically upwards from the mouth of a pit with a velocity of $3g$, and finally reaches the bottom of the pit with the velocity $5g$. Find (1) the depth of the pit; (2) the time of motion.

57. If the units of space and time be each doubled, find the effect on the measures of a velocity and an acceleration.

58. A particle starts from rest at one end of a straight line AB, 364 feet long, under an acceleration of 8 f.-s.; at the same instant another particle is projected from B towards A with uniform velocity. If they meet 64 feet from A, find the velocity of the latter particle.

59. Two balls are thrown up with a velocity of 192 feet per second, one 2 seconds after the other; when and where do they meet?

60. A body is thrown up with a velocity of 100 f.-s., and after 1 second another is thrown up and overtakes the other when at its highest point; find the velocity with which the latter is projected.

61. A particle is started from A towards B with an acceleration of 9 f.-s.; while another particle is moving from B towards A to meet it

with uniform velocity of 120 f.-s., they meet half-way; find the distance between A and B, and the time of motion.

62. A balloon is rising uniformly with a velocity of 10 f.-s., when a stone dropped from it reaches the ground in 3 seconds; find the height of the balloon (1) when the stone was dropped; (2) when the stone reaches the ground.

63. A ball is projected up a plane (inclination = 30°) with a velocity of 64 feet per second. (1.) In what time will the ball return to the point of projection? (2.) What distance does it ascend? (3.) When will the velocity be 256 f.-s.?

64. A balloon is ascending vertically with a velocity of 20 f.-s., and a stone having been let fall from it reaches the ground in 8 seconds. (1.) Find the height of the balloon when the stone was let fall from it. (2.) Find the velocity of the stone when the height of the balloon was 1000 feet.

65. A balloon is rising with a uniform velocity of 50 f.-s., and a stone projected vertically upwards from it with a velocity of 30 f.-s. reaches the ground in 10 seconds; find the height of the balloon (1) when the stone was projected; (2) when the stone reached the ground; find also the greatest height attained by the stone.

66. A balloon ascends with uniform acceleration of 11 f.-s.-s., and at a height of 352 feet a stone is dropped. (1.) How long will the stone continue to rise? (2.) When will it reach the ground? (3.) What was its greatest height?

67. A balloon ascends with uniform velocity for $4\frac{1}{2}$ seconds, and a stone let fall from it reaches the ground in 7 seconds. Find (1) the velocity of the balloon; (2) the height from which the stone fell.

68. A person drops a stone into a well, and after 10 seconds hears the splash; find the depth to the surface of the water, if sound travel at the rate of 1120 f.-s.

69. Assuming the velocity of sound to be 1120 f.-s., find the height of a cliff if the sound of a stone striking the base reach the top in $5\frac{5}{14}$ seconds after it was dropped.

70. A particle is projected vertically up a height of 212 feet with a velocity of 90 f.-s.; after what time must another be projected down from the upper extremity with the same velocity to meet the other half-way?

71. A stone is thrown upwards with a velocity of 150 f.-s., and one second later another stone is projected with a velocity of 200 f.-s. When and where will the stones meet?

72. A stone is projected downwards with a velocity of 50 f.-s. If another stone be projected after it in 10 secs. with a velocity 8 times as great as that of the first, when will it overtake the first?

73. A rocket fired vertically upwards with an initial velocity of 96 f.-s. explodes at its highest point. The interval between the sound reaching the firing-point and a place a quarter of a mile distant from it is one second. Determine the velocity of sound.

74. A balloon is ascending with a velocity of 80 f.-s., and when at a height H a stone is dropped from it. Five seconds later another stone is projected vertically downwards from it with a velocity of 200 f.-s. When will the latter overtake the former? and find the least value of H for this to be possible.

75. A steamer goes due W. at 18 miles per hour; a current is running due S. at 4 miles an hour; a train is going due N. at 60 miles an hour; find the relative velocity of the steamer and train.

76. A falling body describes in the n^{th} second of its fall a space equal to x times the space described in the $(n-1)^{th}$ second; find the whole space described.

77. A particle is projected up a smooth plane inclined at 60° to the horizon with a velocity $\sqrt{gh}$; find (1) how far, (2) how long, it will rise, h being the height of the plane.

78. Two balls are projected vertically upwards with the same velocity u, the second being despatched when the first has attained 3/4ths of its greatest height; at what height will they meet?

79. To a man travelling W. at the rate of 8 miles an hour the wind appears to come from the NW.; when he stands still the wind comes from a point 5° more to the N.; what is the velocity of the wind?

80. If the measure of gravity be 32·2 in the foot-second units, find its measure when the centimetre (=·4 inch) and a second are the units.

81. One ship sailing due N. is 6 miles south of another sailing twice as fast due E.; what will be the shortest distance between them, and when will the bearing of one from the other be NE.?

82. A body is dropped from the top of a tower, and at the same instant two bodies are projected upwards from the ground, one of which meets the falling body at the middle of the tower, and the other after the lapse of half the whole time of descent. Compare the velocities of projection.

83. Two bodies are projected at the same instant, and they are instantaneously under the same acceleration, the initial velocity of one being 5 times that of the other. After $5\frac{1}{2}$ secs. their velocities are 277·1 f.-s. and 677·1 f.-s.; find the initial velocities, and the magnitude of the acceleration.

84. A particle is projected vertically up with a velocity of 102 f.-s. from a certain point O, and 2 secs. later another is projected down from X (vertically above O) with a velocity of 14 f.-s.; find OX, if the latter reach O at the same instant that the former reaches X.

85. A body slides down an inclined plane. It starts with an initial velocity due to the height of the plane. The time of its descent is equal to the time in which it would fall from rest through this height. Find the inclination of the plane.

86. Divide an inclined plane, whose length is l, into n parts, such that the times of describing the successive parts shall be equal.

87. Find the velocity with which a particle should be projected down an inclined plane whose inclination is i, that the time of describing the length (l) may be equal to the time of falling down the height (h).

88. A particle starts from A and slides down a smooth plane AB inclined at 45° to the horizon. Two seconds before it starts another is projected up the plane from B with velocity $\frac{5g}{2\sqrt{2}}$, and meets the first midway between A and B. Find when they meet.

89. The velocities acquired by a body falling vertically down AB and along the inclined planes AC and AD respectively are as the numbers 4, 2, 1, and the time of falling is the same in each case; find the inclinations of the planes to the horizon.

90. With what velocity must a particle be projected downwards so that in t seconds it may overtake another which has already fallen through a distance of a feet?

91. A body has fallen from rest through m feet at the instant another body begins to fall from a point n feet below that which the first then occupies; find the distance traversed by the second before it is overtaken by the first, assuming that the acceleration caused by gravity is constant, but unknown in amount.

92. A stone is projected vertically downwards with a velocity of v, and in t seconds after its projection another stone is projected downwards with a velocity v_1; when will the latter overtake the former?

93. In the last example, what should the second velocity be in order that the first stone may not be overtaken?

94. Two particles slide down the sides AB, AC of a vertical triangle whose base BC is horizontal. Prove that (1) the velocities acquired are equal; (2) the times of describing the sides are proportional to their lengths.

95. Three bullets are fired vertically upwards at the end of three successive seconds; the first with a velocity of 500 f.-s., the second with a velocity of 600 f.-s. Find the initial velocity of the third if the three bullets pass through a plane parallel to the horizon at the same instant.

96. If a particle be projected upwards with a velocity xg, when will it have attained the height of xg, and what will be its velocity at that instant?

Find the least value of x.

97. A particle is moving with uniform acceleration a. If u be the Arith. Mean between its initial and final velocities in describing a space s, and v be the velocity generated during the time of describing that space, show that $uv=as$.

98. A number of beads lie in contact on the horizontal arm AB of a bent wire ABC, BC being vertical, and they are pushed along the wire in the direction AB with uniform velocity. Show that at any time the distances between those which have passed the bend B and are moving vertically are in A. P.

99. Determine that point in the hypothenuse of a right-angled triangle, having its base horizontal, from which the time of a particle's descent down an inclined plane to the right angle is least.

100. A wheel of a carriage is 4 feet in diameter, and makes $3\frac{1}{2}$ revolutions per second. If the carriage be moving at the rate of 30 miles an hour, find the velocity in fixed space (1) of the point of the wheel in contact with the ground; (2) of either of the points in its circumference one yard above the ground. $\left(\pi=\frac{22}{7}\right)$.

101. A particle is let fall from a height h at the same instant that another particle is projected downwards from a point $2h$ above it with the velocity which the first would acquire in falling to the ground. Show that they will reach the ground at the same instant, and compare their final velocities.

102. A particle is thrown vertically up; the time (n seconds) between its leaving a point whose height (h feet) is known and returning to it again is observed; find the velocity of projection.

103. A point has moved from rest with a uniform acceleration, and at the end of t_1, t_2 seconds its velocities are v_1 and v_2 respectively; find the space described in t seconds.

104. If two particles be thrown up from the same point with an initial velocity u, one n seconds after the other, show that they will meet at a height of $\frac{u^2}{2g} - \frac{gn^2}{8}$.

105. $ABCD$ is a square, and E is the middle point of BC; find the resultant velocity of velocities fully represented by AB, AE, and AC.

106. A particle at the centre of a regular pentagon has impressed on it velocities of 4, 3, 2, 2, in the directions of four of the angular points in order; find the resultant velocity.

107. AB is a quadrant of a circle whose centre is O, the radius OB being horizontal; C is a point on the arc of the quadrant, and the angle $BOC=\theta$. Show that the time of falling from A to C is to the time of falling from C to B in the ratio $\sqrt{\cos\frac{1}{2}\theta} : \sqrt{\sin\frac{1}{2}\theta}$.

108. A velocity v, at an inclination θ to the horizon, changes into a velocity $v\cos\theta$ horizontal in time t with a constant acceleration; prove that the acceleration was vertical, and determine its magnitude.

109. PQ is the perpendicular from a point P in the circumference of a circle to a fixed diameter. If P describe the circumference with uniform velocity in 2π seconds, prove that the velocity of Q at any instant is measured by PQ.

110. Three particles are thrown vertically downwards with initial velocities u_1, u_2, u_3 from heights s_1; s_2, s_3 respectively, and they reach the ground simultaneously; prove that $\frac{s_1-s_2}{u_1-u_2} = \frac{s_2-s_3}{u_2-u_3} = \frac{s_3-s_1}{u_3-u_1}$.

111. The distances through which a particle is observed to move from rest in successive seconds are in the proportion of the numbers 1, 3, 5, 7, . . . ; prove that the particle is moving with uniform acceleration.

112. A particle falls through x feet at two different places on the earth's surface, and it is observed that at one place the time of falling is n seconds less, and the velocity acquired is m feet per second greater than at the other; show that if the acceleration caused by gravity at the two places be g_1 and g_2 respectively, then $m/n=\sqrt{g_1 g_2}$.

113. A smooth wire is bent into the form of a right-angled triangle ABC, and is placed with the hypothenuse AB vertical. A ring is projected from C towards B with the velocity which it would acquire by sliding down AC from rest, and another ring begins to move down AB from rest at A; show that the two rings will reach B with the same velocity, and that their times of motion will be as $\tan\frac{1}{2}A : 1$.

114. A Pullman car, 45 feet long, is running at the rate of 60 miles an hour along a straight railway. A bird flying at the rate of 61 miles an hour is seen to pass in through the middle point of one window and to pass out through the middle point of exactly the opposite window. If the bird's path through space is parallel to a diagonal of the floor of the car, what is the width of the car?

115. A man 5 ft. 8 in. high runs from a lamp raised 12 feet at the rate of $6\frac{1}{3}$ miles an hour; find how fast the tip of his shadow moves.

116. If s_t be the distance passed over in the t^{th} sec. by a particle moving from rest with an acceleration a, prove that $2s_t/a$ is always an *odd* integer.

117. A man seated in a railway carriage which is moving at the rate of 30 miles an hour observes a distant object through the opposite window, and notes that it remains in view for fifty seconds. If the breadth of the window be 21 inches, and the distance of the observer's eye from the sides of the window be 8 feet and $7\frac{1}{2}$ feet, find the distance of the object at the moments of its appearance and disappearance from view.

118. Two candles 12 inches and 6 inches long respectively are placed on two stands 1 foot high and 1 foot apart on a table. If the candles burn steadily at the rate of $\frac{3}{20}$ inch per minute, find, in feet per second, the velocity of the extremity of the shadow of the shorter candle cast on the table.

119. In a vertical circle two chords are drawn from the extremity of a horizontal radius subtending arcs θ and 2θ; if the time down the chord of 2θ is n times that down the chord of θ, show that $\sec\theta = n^2 - 1$.

120. A is the highest and B the lowest point of a vertical circle, C a point in the circumference. Two particles are projected with equal velocities, one along CA, the other along CB. The velocity of projection is such that the first particle only just reaches A, and in a time double of that taken by the other to reach B; find the angle ABC.

121. A and B are the highest and lowest points respectively of a vertical circle. From any point P in the circumference a tangent PT is drawn to meet a horizontal line through B in T; show that a particle will run down PT in a time which varies inversely as AP.

122. Two ships A and B, whose rates are 14 and 12 knots respectively, are 6080 feet apart, and they are steaming in opposite directions. A shot is fired from A, and, moving with a mean velocity of 1000 f.-s., strikes B; how far astern of the spot aimed at will the shot strike? (1 nautical mile = 6080 feet.)

123. If v_t be the velocity after t secs. possessed by a particle starting with an initial velocity and moving with uniform acceleration, and a_t be the space described in the same time, show that

$$\frac{a_{t+1}}{t+1}-\frac{2a_t}{t}+\frac{a_{t-1}}{t-1}=v_{t+1}-2v_t+v_{t-1}.$$

124. One particle describes the diameter of a circle with a uniform velocity, and another the semicircle with uniform acceleration in the direction of the tangent; if they start from one extremity of the diameter at the same instant and arrive at the other at the same moment, show that the velocities at the end are in the ratio 1 : π.

125. Three heavy particles are moving at a given instant with velocities v_1, v_2, v_3 respectively up lines inclined at angles α, β, γ to the vertical, and are at that instant in the same horizontal plane. Prove that if at any subsequent time in their motion they are again in a horizontal plane

$$\frac{v_1\cos\alpha-v_2\cos\beta}{\cos^2\alpha-\cos^2\beta}=\frac{v_2\cos\beta-v_3\cos\gamma}{\cos^2\beta-\cos^2\gamma}=\frac{v_3\cos\gamma-v_1\cos\alpha}{\cos^2\gamma-\cos^2\alpha}.$$

126. Two candlesticks, one six inches and the other a foot high, stand on a table two feet apart, and hold candles each a foot long at the moment when lighted. The candles burn at the rate of $\frac{1}{20}$ inch per minute. Find (1) the velocity of the shadow of the top of one candle thrown on the table by the other, and (2) the average velocity of the shadow of the top of the lower candlestick during the whole time which the candles take in burning away.

CHAPTER III.

THE FIRST LAW OF MOTION: MOMENTUM.

80. In the previous Chapters we have dealt with the motion of a point or particle without considering either the quantity of matter moved or what caused the motion.

81. The relations between the quantity of matter moved, the motion produced, and the cause of motion are contained in 'Three Laws of Motion.' These were first enunciated in their present form by Sir Isaac Newton in 1687. The first two were, however, known to Galileo—perhaps were discovered by him—and the Third Law, in some of its many forms, was known to several mathematicians before Newton's time.

82. These laws must be considered as due to actual experience in the shape either of observation or experiment. Their truth is best established *indirectly*: viz., on the assumption of their truth long and complicated computations are made, and the results thus obtained are verified in all cases by observation. The prediction of the time of an eclipse is a good example.

83. **Matter** is something the existence of which we know intuitively, just as we know what *time* is and what *space* is; but for our present purpose it may be defined to be anything which can have its motion changed by some cause. (*See* Introduction.)

The *quantity of matter* contained in a body is technically called its **Mass.**

The mass of a body is invariable, *i.e.* it remains the same wherever the body may be.

84. Newton established by experiment the extremely important fact that a practical measure of the Mass of any body is its Weight (*see* Arts. 95, 110, and 126), and therefore that in a balance equal masses will always counterpoise each other.

85. Since Mass is a thing which may be measured, it is necessary to decide on a *unit* by which it can be measured. Most nations have their own Units of Mass, known in popular language as *Standard Weights*

86. The term *Weight* denotes a *force*, as we shall see presently (*see* Arts. 95 and 110); viz., the force of the earth's attraction on a mass; and some confusion may, and in fact does, arise from this double use of a word. The student ought to bear in mind that a national 'Standard Weight' is intended to be a test of mass. It is *exclusively* used in commerce to measure out a definite *quantity of some commodity—i.e.* of matter, and is not designed to determine the quantity of matter which shall be attracted by the earth with a given force.

These national standards ought evidently to be of some material not liable to waste in careful handling, or through the corroding action of the atmosphere. Platinum, or a mixture of platinum and iridium, is found to be most suitable for this purpose.

87. The **British Unit of Mass** is the quantity of matter contained in a definite piece of platinum, preserved in the Exchequer Office, authorised copies of which are kept in the Mint and other places (Introduction). This is known as a *Pound Avoirdupois*, or an *Imperial Pound.*

The French Unit of Mass is called a Kilogramme, and is equal to 2.204 pounds.

88. Matter has an innate incapacity to alter its state of rest or of uniform motion in a straight line, and therefore every portion of matter, so far as it is isolated from all external influences, remains at rest or moves uniformly in a straight line. When we speak of the 'Inertia of matter,' we mean that matter is perfectly passive, and at once responds to any the slightest force which acts upon it.

89. If, therefore, a body previously at rest be found to be moving, or, if previously moving in a straight line, be found to have its velocity changed in any way, then we infer that some cause has produced that change. And this cause is termed *Force.*

We are now in a position to state the First Law of Motion.

90. First Law.—'*Every body continues in its state of rest or of uniform motion in a straight line except in so far as it may be compelled by force to change that state.*'

This law expresses the fact that matter is indifferent to rest or motion; or, that a state of uniform rectilinear motion is as much a proof as a state of rest that a body is not acted on by force.

The law affords a definition of Force.

91. DEF.—**Force** *is any cause which alters or tends to alter a body's state of rest or of uniform motion in a straight line.*

92. Of course we can never completely test this law by experiment, because we can never isolate a body from the influence of external force. But, by reducing the action of external force as much as possible, we can approximate to the conditions required by the law. The following **illustrations** will make this clear:—

1. When a body previously at rest is found to have changed its position, the motion can always be traced to the action of some force.

2. A curling-stone will go further on ice than on a road, and the smoother the ice the further it will travel; and also the more nearly will the velocity of the stone be uniform and in a straight line.

3. A pendulum will swing longer in an exhausted receiver than in the open air, and the greater the degree of exhaustion the longer it will continue to swing.

4. When a train is approaching a station, if the speed be considerable and the rails very smooth, it is found difficult to apply the brake with effect.

5. When a carriage turns round a sharp curve there is a tendency for any occupant to shoot on in the straight line previously pursued.

6. When a person stamps to get rid of mud or snow, the inertia of matter tends to detach it from the boot.

7. When a train pulls up suddenly, a passenger, as a rule, sustains a jerk, from the muscular effort made to check the onward motion of his body.

8. A person sitting at the back of a dog-cart at rest is *apt to be left behind* if the horse make a sudden start forward.

9. The following is a good illustration, and will be better understood on reading the subject a second time.

When a train is moving on a horizontal railroad, a constant expenditure of coals or steam is required to maintain a constant velocity. The opposing resistance caused by the air and friction is balanced by a constant force, and the train is in the same condition as if it were acted on *by no force at all*, and hence, as the law asserts, it will move uniformly.

* **93.** There are three elements specifying a Force, viz.:—(1) its Point of Application; (2) its Direction; and (3) its Magnitude. These must be known before a clear notion of the force in question can be formed. It is evident that as a finite straight line is also determined by the same three elements, any given force may also be fully represented by a straight line. (*See* Article 156.)

94. Forces are known by different names, such as Pressure, Tension, Friction, Electrical, Chymical. Some of these will require detailed explanation later on.

95. The *Law oj Gravitation* asserts that every particle of matter in the universe attracts and is attracted by every other particle. In consequence of this law the earth tends to draw all masses towards itself. The force which the earth exerts on the mass of a body is called the *Weight* of that body. The student will therefore bear in mind that *weight* means a *force.* (*See* Article 84.)

96. Now let us suppose that there are three bodies of equal mass, and that all three are moving with the same velocity; then we say that they possess equal 'quantities of motion.' If we conceive two of these bodies to be joined and to become one body, the velocity remaining as before, we are plainly justified in saying that the 'quantity of motion' possessed by this larger body is double the amount possessed by the other.

Again, if there are two bodies of equal mass, and one of these has a velocity twice as great as that of the other, then it is evident that the 'quantity of motion' possessed by the first is double the quantity possessed by the other.

97. The 'quantity of motion' must be distinguished from 'velocity.' These have the same relation to one another as 'quantity of heat' and 'temperature.' And as we would say that 100 gallons of water at a temperature of 160° F. have more heat but a less temperature than a cup of boiling water, so we can say in like manner that a locomotive moving at the rate of 40 miles an hour has more Motion but less Velocity than a bullet shot with a velocity of 900 feet per second.

98. The 'quantity of motion' of any body depending, therefore, on its mass and its velocity, it is easy to see that, if certain units of mass and velocity be selected, the numerical measure of the 'quantity of motion' will be found by multiplying together the measures of the mass and the velocity.

***99.** The term **Momentum** is applied to the *product of the mass of a body and its velocity.*

100. It is necessary to select a Unit of Momentum. As the British unit of mass is a pound (Art. 87), and our unit of velocity is a velocity of a foot per second (Art. 6), so our **British Unit of Momentum** will be the '*Quantity of Motion possessed by a pound of matter moving at the rate of a foot per second.*'

If a body contain m pounds of matter and be moving with a velocity v feet per second, it will possess mv units of Momentum, and therefore its Momentum $= mv$.

And, generally, if in any system of units a body contain m units of mass, and be moving with v units of velocity, then we shall have its Momentum $= mv$.

NOTE.—There is as yet no widely recognised name for the 'Unit of Momentum.'

Example i.—A mass of 10 lbs. is moving with a velocity of 18 f.-s.; what momentum does it possess?

$$\text{Momentum} = mv.$$

$$\therefore \qquad \text{,,} \qquad = 10 \times 18 = 180 \text{ units of momentum.}$$

Example ii.—A mass of 20 lbs. is falling freely at a place where the acceleration caused by gravity is 32·2; find the momentum after 3 and after 10 seconds respectively.

(1) *After* 3 *seconds*, $v = u + at = 32{\cdot}2 \times 3 = 96{\cdot}6$;

$\therefore$ Momentum $= mv = 20 \times 96{\cdot}6 = 1932$ units of momentum

(2) *After* 10 *seconds*, $v = u + at = 32{\cdot}2 \times 10 = 322$;

$\therefore$ Momentum $= 20 \times 322 = 6440$ units of momentum.

EXAMPLES—XX.

1. What is the momentum of a mass of 6 lbs. moving with a velocity of 25 f.-s.?

2. A mass of 18 lbs. has moved from rest with an acceleration of 16 f.-s.-s. What is its momentum (1) after 8 seconds, (2) after it has described 576 feet?

3. An anchor whose mass is $4\frac{1}{2}$ tons falls 16 feet; with what momentum does it reach the water?

4. The masses of two trains are 40 and 50 tons respectively, and they are travelling with speeds of 40 and 50 miles an hour respectively; compare their momenta.

5. The mass of a boat and its contents is $3\frac{1}{2}$ tons, and it is moving with a velocity of $7\frac{1}{2}$ miles an hour; find the momentum.

6. A body whose mass is 5 cwt. is moving with a velocity of $22\frac{1}{2}$ miles an hour; through what height must the body fall to acquire an equal momentum?

7. Find the momentum of a mass of 25 lbs. when it has fallen for 6 seconds.

8. Find the increase of momentum acquired by a mass of 100 lbs., falling freely, between the 4th and 9th seconds.

9. A mass of 30 lbs. is travelling with an acceleration of 10 f.-s.-s. In one position its velocity is 40; find its momentum after describing another 200 feet.

10. A body having a momentum of 100 describes 45 feet in 3 seconds with uniform velocity; find its mass.

11. Through what space must a mass of 60 lbs. fall that its momentum may be 1000?

12. A mass of 100 lbs. is thrown vertically upwards with a velocity of 256 f.-s.; find its momentum after 1, 4, 8, 10 seconds respectively.

13. Compare the momenta of a mass of 100 lbs. moving with a velocity of 1000 f.-s., and a mass of 1 lb. moving with a velocity of 30 feet per minute.

14. If the units of length, time, mass, be 176 yards, 1 minute, and a hundredweight of matter respectively, what is the measure of the momentum of a train of 100 tons, moving with a velocity of 40 miles an hour?

15. A mass of 20 lbs. moves from rest through a space of 200 feet with acceleration 16 f.-s.-s.; compare its final momentum with that acquired by a mass of 30 lbs. falling from a height of 1024 feet.

CHAPTER IV.

THE SECOND LAW OF MOTION.

101. SINCE the effect of a Force is to produce motion, it will appear reasonable that the Magnitude of the force should be estimated by the 'quantity of motion' it is capable of producing; and the Second Law of Motion asserts that this is the case. momentum

102. Second Law.—'*Change of motion is proportional to the force applied, and takes place in the direction of the straight line in which the force acts.*'

NOTE.—'Motion,' in this statement, means 'quantity of motion,' *i.e.* Momentum. (*See* Art. 99.)

103. Let us first notice the very important facts *implied* in the statement of this Law.

(*a*) It speaks only of the *change* of momentum, *i.e.* it takes no notice whatever of any motion already possessed by the body when a force begins to act on it. We infer therefore, by this Law, that a force will produce exactly the same change of momentum whether the body is at rest or in motion at the instant the force begins to act on it.

(*b*) It says nothing which restricts it to *one* force. We infer therefore, by this Law, that if *any number* of forces act on a body, *each* will produce exactly the same change of momentum which it would produce if it acted alone on the body.

104. The student will remember that we do not *prove* the Laws of Motion by experiment. We *assume* that they are a correct interpretation of facts in nature. On this assumption extremely complicated work is undertaken, and remote consequences are arrived at, and, as has been said (Art. 82), every such computation, when verified, will afford a fresh proof, though an *indirect* one, of the truth of the laws which form the basis of our reasoning.

105. The following **illustrations** of the statement that a force will always produce its full effect in its own direction may be given :—

1. A ball thrown along the deck of a steamer will go the same distance on the deck whether the ship be at anchor or in motion.

2. A stone dropped from a mast-head will strike the deck at the foot of the mast, whether the ship be at rest or in motion.

Thus, in the latter case, the vertical motion caused by gravity has not been changed by, nor has it changed, the horizontal motion given to the stone by the ship.

3. If any number of balls be projected instantaneously from the summit of a cliff with different velocities in a horizontal direction, they will reach the water simultaneously. Thus the horizontal motion does not interfere with the action of gravity producing its full effect.

4. If a traveller, in a railway carriage having rapid motion, throws a body upward, it will fall into his hand as if the train were at rest.

5. A circus-rider intending to jump through a hoop will spring *vertically ;* his previous horizontal motion, being the same as that of the horse, will not be altered, and thus he will alight on the horse again.

106. The Second Law of Motion asserts, and we therefore assume as a fact, that a force is proportional to the change of momentum which it can produce.

We infer from the statement (a) in Article 103, that as long as a force continues to act on a body it is producing momentum in addition to the momentum already produced—in other words, the force will be proportional to the *rate of change of momentum* which it is capable of producing.

107. Now if a force act on a single body, since the mass of that body is constant, the rate of change of momentum will evidently depend on the *rate of change of velocity*, that is on the *acceleration*, produced. (*See* Art. 43.)

108. We may exhibit this extremely important fact as follows:—Let F denote the magnitude of a force acting on a body, the mass of which is m, and let a be the acceleration.

Then the Second Law asserts that—

$F \propto$ rate of change of momentum;

$\therefore F \propto$ rate of change of mv;

$\therefore F \propto$ rate of change of v, if m be constant,

but a is the rate of change of v;

$\therefore F \propto a$, if m be constant;

And also $F \propto m$, if a be constant

$\therefore F \propto ma$, when m and a both vary.[1]

Now uniform acceleration in any direction is measured by the velocity generated in a second. (Art. 51.)

We may therefore say that a Force is proportional to the Momentum which it produces in a second.

109. *First.* $F \propto a$, if m be constant.

We infer, therefore, that if different forces act on the same mass the forces will be proportional to the acceleration produced, and therefore to the velocities produced in a second.

Thus the velocities acquired in a second by the same mass falling freely at different places, enable us to compare the *force of gravity* at those places.

We may define *Equal Forces* as those which can produce in the same mass the same acceleration.[2]

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[1] This Theorem in Variation is proved in any work on Algebra.

[2] We shall see presently (Art. 124) that whatever be the mass of the body acted on, a given force will produce in it the *same change of momentum* in a second.

110. *Next.* $F \propto . m$, if a be constant.

We infer, therefore, that if different masses acquire *the same acceleration*, the forces acting on them are proportional to the masses on which they act.

Now this is the case when bodies fall freely in a vacuum at any place. All bodies, whatever they are composed of, acquire the same acceleration, viz. the value of g at the place. We must therefore conclude that the *Weight* of any body at any place is proportional to the mass of the body.

This most important result was verified by the careful experiments of Newton, and it is so well established that no other method of comparing masses, than that of *comparing their weights* at the same place, is ever made use of either in commerce or science. (*See* Art. 125.)

111. We conclude, then, that by the Second Law of Motion,

$$F \propto ma,$$

$$\text{and} \therefore F = k . ma.$$

Now by a suitable arrangement of units we can make $k = 1$, and then the above equation of variation may be written as an equation, thus,

$$F = ma.$$

112. What is the most suitable arrangement of units to make in order that k may have this convenient value?

We have already selected a foot and a second as our units of space and time, and have decided that the unit of acceleration shall be the 'velocity of a foot per second added every second.' (*See* Art. 50.)

Now the equation $F = ma$ restricts us in the units of force and mass to this extent, that if either unit be chosen, the equation determines the other unit; the unit of acceleration having been already decided on. To show that the equation does so, let us select the Units of Force and Mass in turn.

113. Let us first select the *Unit of Force.*

Let this force act on a mass so chosen that the acceleration produced is the unit acceleration.

Then the above equation gives us—

$$1 = m \times 1 \,; \quad \therefore m = 1.$$

Therefore the *Unit of Mass* must be that amount of matter in which the unit of force will produce the unit acceleration.

114. Let us next select the *Unit of Mass.*

Let a sufficient amount of force act on this quantity of matter to produce the unit acceleration.

Then the above equation gives us—

$$F = 1 \times 1 \,; \quad \therefore F = 1.$$

Therefore the *Unit Force* must be that amount of force which, acting on the unit mass, will produce the unit acceleration.

115. We have now to consider the two alternatives thus offered to us.

Suppose we select our Unit of Force, and let a *Pound Weight* be the unit chosen.

A Pound Weight is the force of the earth's pull
on the mass of a pound.
If a pound of matter be allowed to fall freely,
then a Pound Weight is the force acting on it,
and the acceleration produced is g.
$\therefore$ A Pound Weight will produce in 1 lb., g units of acceleration.
$\therefore$ to produce only the unit of acceleration,
it must act on mass of g lbs.

Hence if Unit of Force = a Pound Weight, we must take *g lbs. of matter* as the Unit of Mass.

Or, using the equation $F = ma$,
Let $F = 1$, and let the acceleration $= g$.
Then $1 = mg$,

$$\therefore m = \frac{1}{g} \cdot$$

That is, the Quantity of matter acted on when an acceleration of g is produced by the action of a Pound Weight is $\frac{1}{g}$th of the Unit of Mass;

but we require only an acceleration $= 1$;

$\therefore$ Unit of Mass $= g$ lbs. of matter.

116. All bodies having weight, it has been usual to compare forces with the Weight of a body, and to speak of a force as so many Pounds Weight. When forces are thus expressed in *terms of weight*, we are said to use *Gravitation Measure.*

117. There is, however, a double objection to making a Pound Weight the unit of force.

1. We have a Unit of Force which is not constant.

A Pound Weight is greater as we advance from the Equator towards either Pole. A 'Pound Weight' is never definite, and therefore of no scientific value, unless the locality be specified where the measurement was made.

2. And the Unit of Mass is not constant, depending as it does on the local value of g.

118. Now Gravitation Measure will answer perfectly so long as we are called on to compare forces at the same place only. It will also serve practically where strict accuracy is not important. In Statics this method of measuring force may be used with advantage—and is, in fact, nearly always used; but for strictly scientific purposes, such as comparing the earth's horizontal magnetic force in different places, recourse must be had to another method.

Professors Thomson and Tait directed attention, in the first edition of their *Treatise on Natural Philosophy*, to Newton's method of measuring force; and to this method the student's attention will now be drawn.

119. Let the *Unit of Mass* be first chosen, and let this unit be an *imperial pound* of matter.

Then we have seen that the (*British*) *Unit of Force* will be that amount of force which will produce in a pound of matter the unit acceleration. (Art. 114.)

The amount of force required to produce a stated quantity of momentum in a given time is perfectly definite (Art. 106), and is quite independent of locality. Hence the British Unit of Force is invariable.

On a similar principle, the unit of force in the C. G. S. system of units being the amount of force which, acting on a gramme, will produce each second a change of velocity of a centimetre per second, is also invariable.

These units are known as *Absolute* or *Kinetic* units of force.

120. There is a twofold advantage in first selecting the unit of mass:—(1) We secure an invariable unit of mass; (2) we deduce a constant unit of force.

The British Kinetic unit of force is called a **Poundal.**[1]

The C. G. S. Kinetic unit of force is known as a **Dyne.**

NOTE.—A foot contains 30 4797 centimetres, and a pound is equal to 453·59 grammes. A Poundal is equal to 13825·28 Dynes.

121. *To find the value of a Poundal in terms of weight.*[2]

1 Pound Weight will produce in 1 lb. mass an acceleration g;
1 Poundal „ 1 lb. „ „ 1;

$$\therefore 1 \text{ Pound Weight} = g \text{ Poundals};$$

$$\therefore \frac{1}{g} \text{ Pound Weight} = 1 \text{ Poundal}.$$

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[1] By Professor James Thomson.

[2] Weight has been aptly called the *coinage* of force.

By very careful experiments, the average acceleration due to the force of gravity for the whole of the British Islands is found to be nearly 32·2.

Hence 32·2 may be taken as the number of Poundals equivalent to the Weight of a Pound in Great Britain.

A Pound Weight = 32·09 Poundals at the Equator.
,, = 32·18 ,, Greenwich.
,, = 32·25 ,, the Pole.

If g be taken as 32, we can take the force of a poundal as equal to the weight of ½ oz.

122. Since 1 Pound Weight = g Poundals, it is evident that we can express the Kinetic measure of a force in terms of Gravitation measure at any place, by dividing the number of Poundals by the local value of g; and we can always change the Gravitation measure of a force at any place into Kinetic measure, by multiplying the number of Pounds Weight at that place by the local value of g.

Example i.—Express in poundals the value of 3 pounds weight at the Equator.

1 pound weight = 32·09 poundals;
∴ 3 pounds weight = 96·27 poundals.

Example ii.—How many pounds weight at Greenwich will represent a force of 640 poundals?

32·18 poundals = 1 lb. weight at Greenwich.

$$\therefore 1 \text{ ,, } = \frac{1}{32{\cdot}18} \text{ ,, ,,}$$

$$\therefore 640 \text{ ,, } = \frac{640}{32{\cdot}18} \text{ ,, ,,}$$

$$= 19{\cdot}88 \text{ lbs. weight nearly.}$$

123. This Absolute method of measuring force was first indicated by Newton, and was first practically introduced by Gauss. Newton never spoke of a force as being equal to so many pounds weight or tons weight; in other words, the Gravitation Measure of force receives no countenance from him.

124. The equation $ma = F$ is the **Fundamental Equation of Dynamics,** and the student must bear in mind that it expresses the important relation—'*Number of Units of Mass* × *Number of Units of Acceleration* = *Number of Units of Force acting.*'

Thus we see that the *proper measure of a force* is 'the rate at which it can produce momentum.'

And, practically, a finite Force is measured by the change of momentum which it can produce in a second.

125. Special Case *of the equation* $ma = F$.

If m be allowed to fall freely the acceleration is g, and the force acting is the *Weight of m.* Calling this force W, we have

$$mg = W.$$

If two bodies, whose masses are m_1 and m_2, be allowed to fall freely at the same place, they will move with the *same* acceleration (g) under the action of their weights w_1 and w_2 respectively.

$$\text{Hence } m_1 g = w_1, \text{ and } m_2 g = w_2;$$

$$\therefore \frac{m_1}{m_2} = \frac{w_1}{w_2};$$

and therefore the masses are proportional to their weights at the same place. (*See* Art. 110.)

126. It is a fact discovered by observation, and verified by the carefully conducted experiments of Newton, that bodies of equal mass (whatever be the matter they are composed of—gold, silver, lead, glass, sand, salt, wood, water, wheat)—placed in the same position relative to the earth, are attracted with an equal force towards the centre. The student will notice that a merchant using a balance and a set of standard weights (≡Masses, see Art. 86) would give his customers the same quantity of a commodity however the earth's attraction might vary, his instrument depending on the weights of *constant masses*; but it is otherwise with a merchant using a spring balance correctly adjusted for a certain place (London, for example). With this instrument, which depends on *constant forces,* he would defraud his customers in high latitudes and himself in low latitudes.

127. By Article 125 we have—

$$mg = W; \text{ and } \therefore\ m = \frac{W}{g}.$$

That is, the number of Units in the Mass of a body is equal to the Weight of the body at any place divided by the local value of g.

128. If, in any example, a mass and a force be expressed in the same terms—*e.g.* a force of P pounds weight and a mass of Q pounds—the student may apply the equation $ma=F$ to either the Gravitation or the Kinetic method of measuring force.

(1.) If he decide to work by *Gravitation* measure of force, the force acting $=P$ pounds weight, and his mass $=\frac{Q}{g}$ units of mass. (Arts. 115 and 127.)

(2.) If he prefer to use the *Kinetic* measure of force, then his mass $=Q$ lbs., and his force $=Pg$ poundals. (Art. 121.)

Several examples fully worked will, it is expected, remove any ambiguity. Take $g=32$.

Example i.—A force of 5 pounds weight acts on a mass of 48 lbs.; find the acceleration.

We have $ma=F$.

(1) Using *gravitation* measure, then $F=5$, and $m=\frac{48}{g}$;

$$\therefore \frac{48}{g}\cdot a = 5; \quad \therefore\ a = \frac{5g}{48} = \frac{10}{3} \text{ f.-s.-s.}$$

(2) Using *kinetic* measure, then $m=48$, and $F=5g$;

$$\therefore 48a = 5g; \quad \therefore\ a = \frac{5g}{48} = \frac{10}{3} \text{ f.-s.-s. as before.}$$

Example ii.—A mass of 30 lbs. is acted on by a force which produces in a second a velocity of 14 f.-s.; find the magnitude of the force.

We have $ma=F$.

(1) Using *gravitation* measure, $m=\frac{30}{g}$; $a=14$.

$$\text{Then } \frac{30}{g}\times 14 = F; \quad \therefore\ F = \frac{30\times 14}{32} = 13\tfrac{1}{8} \text{ lbs. weight.}$$

(2) Using *kinetic* measure, $m=30$, $a=14$.

$$\text{Then } 30\times 14=F;$$

$$\therefore\ F=420 \text{ poundals.}$$

Example iii.—A mass of 24 lbs. having a velocity of 100 f.-s. is resisted by a force of 1 lb. weight; how long will it continue to move?

$$ma=F.$$

Using *kinetic* measure, $m=24$; $F=1.g$;

$$\therefore\ 24a=1.g;$$

$$\therefore\ a=\frac{32}{24}=\frac{4}{3}.$$

$$\text{Then } v=u+at;$$

$$\therefore\ 0=100-\frac{4}{3}t;$$

$$\therefore\ t=75 \text{ seconds.}$$

Example iv.—An engine moving at the rate of 30 miles an hour has the steam shut off, and a brake power $=\frac{1}{320}$ weight of the engine is applied; in what time will the engine be brought to rest?

Using *gravitation* measure, let W = weight of the engine.

$$\text{Then } \frac{W}{g}=\text{mass} \quad ,, \quad ,,$$

$$\frac{W}{320}=\text{force acting.}$$

$$\text{We have } ma=F.$$

$$\text{Then } \frac{W}{g}.a=\frac{W}{320};$$

$$\therefore\ a=\frac{32}{320}=\frac{1}{10} \text{ f.-s.-s.}$$

A velocity of 30 miles an hour = a velocity of 44 f.-s.

$$\text{Then } v=u+at;$$

$$\therefore\ 0=44-\frac{1}{10}.t;$$

$$\therefore\ t=440 \text{ seconds.}$$

Using *kinetic* measure, let m = mass of the engine.

$$\text{Then } mg=\text{weight} \quad ,, \quad ,,$$

$$\therefore\ \frac{mg}{320}=\text{force acting.}$$

$$\text{Then } ma=F;$$

$$\therefore\ ma=\frac{mg}{320};$$

$$\text{or } a=\frac{32}{320}=\frac{1}{10} \text{ f.-s.-s. as before.}$$

EXAMPLES—XXI.

1. A force produces in a mass of 40 lbs. an acceleration of 10 f.-s.-s.; find the gravitation measure of the force.

2. A force produces in a mass of 6 lbs. an acceleration of 12 f.-s.-s.; find the kinetic measure of the force.

3. A railway carriage whose mass is a ton is pushed by a force equal to the weight of 112 lbs. along smooth rails; find the acceleration.

4. A force equal to 3 lbs. weight acts on a mass of 16 lbs., and a force of 4 lbs. weight acts on a mass of 20 lbs.; which body will acquire the greater velocity in a given time?

5. A force of 15 lbs. weight acts on a mass of 300 lbs.; find the space described from rest in 20 seconds.

6. A mass of 5 tons is acted on by a constant force of 200 lbs. weight; find the space described from rest in 7 seconds.

7. Find the force which in ½ mile would stop a train whose mass is 50 tons moving at the rate of 24 miles an hour.

8. A mass of 1 ton under the action of a constant force describes 180 ft. from rest in 15 seconds; find the force in gravitation measure.

9. In what time will a force of 5 lbs. weight move a mass of 10 lbs. a distance of 50 feet on a smooth plane? What will be the velocity of the body at the end of 12 seconds?

10. If a force of 32 poundals act on a body for 1 minute, and produce a velocity of 400 f.-s., find the mass of the body.

11. Through what distance must a force of 3 lbs. weight act on a mass of 50 lbs. to increase its velocity from 30 f.-s. to 45 f.-s.?

12. A mass of 20 lbs. thrown along ice with a velocity of 40 f.-s. comes to rest in 6 seconds; find the magnitude of the resistance.

13. A mass of 8 lbs. is moved through 2½ feet in the first second of its motion; express the measure of the force in poundals.

14. How long must a force of 3 oz. weight act on a mass of 12 oz. to produce in it a velocity of 40 f.-s.?

15. If a mass of 25 oz. under the action of a constant force describe a distance of 320 feet in 10 secs., find the value of the force in poundals.

16. A mass of 8 lbs. describes 5 feet in the first second of its motion; find the value of the force acting on the body in gravitation units.

17. What force (in poundals) must act on a mass of 48 lbs. to increase its velocity from 60 f.-s. to 90 f.-s. while the body passes over 120 feet?

18. A ball whose mass is 12 lbs. is thrown along ice with a velocity of 96 f.-s., and is brought to rest in 10 secs.; find the resistance in terms of pounds weight.

19. Find the resistance (in poundals) when a body whose mass is 20 oz., projected along a rough table with a velocity of 48 f.-s., is brought to rest after 5 secs.

20. A body whose mass is 50 lbs. is acted on by a force for 5 secs. only, the body then describes a distance of 60 feet in 2 seconds; find the force and express its measure in poundals and pounds weight.

21. A train quickens its speed uniformly from starting, and in 3 minutes describes a mile; compare the force exerted by the engine with the weight of the train.

22. What constant force will lift a mass of 50 lbs. vertically upwards through 200 feet in 10 seconds? and find the velocity of the body at the end of that time.

23. What pressure, acting for 10 seconds, will produce in a mass of 36 lbs. a velocity of 120 f.-s.?

24. A mass originally at rest is acted on by a force which in 1/368th part of a second gives it a velocity of 5½ inches per second; find the ratio between the force and the weight of the mass.

25. An engine produces in a train, whose mass (including that of the engine) is 60 tons, a velocity of 5 miles an hour in 5 minutes. If the same engine produce in another train a velocity of 7 miles an hour in 10 minutes, find the mass of the latter.

26. Two masses, $3m$ and $5m$, are acted on by forces which produce in their motions accelerations of 7 and 9 respectively. Compare forces, and also amounts of force expended on the two bodies in any the same time.

27. Two forces whose magnitudes are in the ratio 3 : 5 act on two bodies and communicate velocities 5 and 11 in 3 seconds; compare the masses of the bodies.

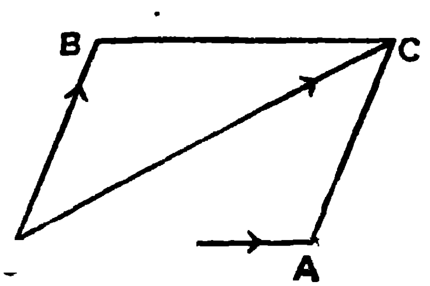
28. Two bodies are acted on by equal forces, and starting from rest describe the same space, the one in half the time that the other does. Compare their final velocities and momenta.

129. We have already seen that when any number of forces act simultaneously on a body, each produces the same rate of change of momentum as if it acted alone (Art. 103 *b*), and also that a force is measured by the change of momentum which it can produce in a second (Art. 124).

Now if the mass moved be constant, the force acting will be proportional to the velocity produced in a second; we can therefore assert that if several forces act on the same body, each force will be proportional to the velocity produced by it in a second, and will be represented in direction by that velocity.

PARALLELOGRAM OF FORCES.

130. STATEMENT.—*If two Forces acting on a body be represented in magnitude and direction by two adjacent sides of a parallelogram, their Resultant is represented in magnitude and direction by the diagonal which passes through their intersection.*



If a body be acted on by two forces represented in magnitude and direction by OA and OB, then each force will produce in a second a velocity proportional to itself in magnitude and direction (Art. 129).

Therefore OA and OB will represent the velocities produced in a second by the two forces.

The resultant of these velocities $\equiv OC$.

The Resultant Force must be proportional to the resultant velocity.

∴ OC will represent both in magnitude and direction the Resultant of the two forces which are represented in magnitude and direction by OA and OB. (Q.E.D.)

131. From this Proposition we may deduce the '**Triangle of Forces**' and the '**Polygon of Forces,**' as the corresponding propositions were established in Articles 20, 21; and by the methods explained in Articles 14-27, we can compound and resolve *Forces* in exactly the same way as we compound and resolve *Velocities*, and hence the proposi-

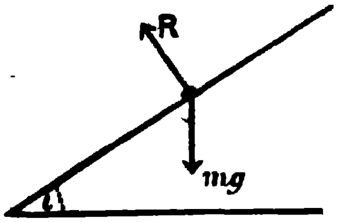
tions in Articles 17-30 hold good when Forces are substituted for Velocities. (*See* Arts. 59, 60.)

132. The case of the *equilibrium* of a number of forces acting at a point is also deducible at once from this way of treating the subject. If we introduce a Force equal and opposite to the Resultant, it will cause a momentum equal and opposite to the momentum produced by the Resultant, and will, therefore, produce a condition of things where the body experiences no momentum, which is, evidently, the condition of equilibrium.

Thus the Second Law of Motion, as Newton perceived, 'contains the most philosophical foundation for the Static as well as for the Kinetic branch of the Dynamic Science.'[1]

'It is manifest that when a system of forces is exactly balanced, and is equivalent to no force at all, the forces will also be balanced if they act in the same way on any other material system, *whatever be the mass* of that system. This is the reason why the consideration of mass does not enter into statical investigation.'[2]

133. We are now in a position to find the Acceleration when a body slides down a smooth inclined plane under the action of gravity.



Let m denote the mass of the body, and a the acceleration in the direction of the plane.

The forces acting on m are its Weight and the Pressure of the plane on m, and this we denote by R.

Resolve the forces acting on m along the plane, and at right angles to it.

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[1] Thomson and Tait, *Natural Philosophy*, vol. i. p. 245.

[2] Clerk Maxwell, *Matter and Motion*, p. 46.

Consider, first, the latter set of components:—

Let k = acceleration at right angles to the plane;

$\therefore mk$ = force acting in this direction;

$\therefore mk = R - mg \cos i$.

But the motion at right angles to plane = 0.

$\therefore 0 = R - mg \cos i$ . . . . (1.)

Consider, next, the components along the plane:—

Let a = acceleration along the plane;

$\therefore ma$ = force acting along the plane;

$\therefore ma = mg \sin i$;

$\therefore a = g \sin i$ . . . . (2.)

This result (2) was assumed in Article 65.

134. We have seen (Art. 124) that if a force act on a body free to move under the action of that force, the equation $ma = F$ connects the mass of the body, the magnitude of the force, and the acceleration produced.

Now we know $v = u \pm at$, by Articles 61 and 64, Note 2. Eliminate a from this equation, and from the equation $ma = F$.

$$\therefore v = u \pm \frac{F}{m}.t; \quad \therefore \pm (mv - mu) = Ft.$$

Now mu will measure the momentum possessed by the body at the instant F begins to act on it, and mv its momentum after t seconds.

If the force act in the direction in which the body is moving, we must take the *positive* sign in this equation; if in the opposite direction, then we take the *negative* sign.

(1.) If F be acting in the direction of the previous motion, then— $mv - mu = Ft$;

i.e. the *Momentum gained* measures the amount of force expended in the time.

(2.) If F be acting in the direction opposite to that of the previous motion, then—

$$+mu - mv = Ft;$$

i.e. the ***Momentum lost*** measures the amount of force expended in the time.

(3.) If the body start from rest under the action of a force, then $u=0$, and $mv-0=Ft$.
i.e. the ***Momentum produced*** measures the amount of force expended in the time.

(4.) If the velocity of a body be destroyed by the action of a force resisting motion, then—

$$v=0, \text{ and } 0+mu=Ft.$$

$\therefore$ the amount of force expended in the time is measured by the ***Momentum destroyed.***

Example i.—A mass of 100 lbs. moving with a velocity of 96 f.-s. is retarded by a force of 8 lbs. weight; how long will the body move?

$$\text{Momentum}=Ft;$$
$$\therefore\ 100\times 96=8g.t;$$
$$\therefore\ \frac{9600}{8\times 32}=t;$$
$$\therefore\ \text{Time}=37\tfrac{1}{2}\text{ seconds.}$$

Example ii.—How long must a force of 10 lbs. weight act on a mass of 128 lbs. to increase its velocity from 50 f.-s. to 110 f.-s.?

$$mv-mu=Ft;$$
$$\therefore\ 128\,(110-50)=10\,g.t;$$
$$\therefore\ t=24\text{ seconds.}$$

EXAMPLES—XXII.

1. By how much is the momentum of a mass of 40 lbs. falling freely, increased between the 3rd and 6th seconds?

2. A mass of 14 lbs. is moving on ice with a velocity of 80 f.-s., and comes to rest in 30 seconds; what is the force of friction?

3. A mass of 100 lbs. moving with a velocity of 136 f.-s. is retarded by a force of 17 lbs. weight; how long will it move?

4. If a friction of $\frac{1}{10}$ of the weight retard a mass of 20 lbs. moving with a velocity of 60 f.-s., in what time will it bring the mass to rest?

5. If a mass of 20 lbs. moving with a velocity of $\frac{32}{3}$ f.-s. be subjected to a force of friction which in $3\frac{1}{3}$ seconds brings it to rest, find the magnitude of the force.

6. A mass becomes subject to a resistance of ·15 pounds weight per ton; in what time will its speed decrease from 40 to 30 miles an hour?

7. How long will a mass of 150 lbs. moving with a velocity of 30 f.-s. move against a resistance of 10 pounds weight?

8. A train of 100 tons mass, moving at the rate of 30 miles an hour, is brought to rest in $4\frac{1}{2}$ minutes; find the retarding force.

9. A mass of 25 lbs. having fallen through 400 feet, what amount of force will stop it in 5 seconds?

10. An engine exerts a steady pull of 10,080 poundals on a mass of 81 tons. In what time will the body pass over $17\frac{1}{22}$ miles from rest on a smooth horizontal plane?

11. An engine can exert a steady pull equal to the weight of $2\frac{1}{28}$ tons. In what time will it get up in a train of 160 tons a velocity of 45 miles an hour from rest, the resistances being equivalent to the weight of 12 lbs. per ton?

12. A mass of 112 lbs. is acted on by a horizontal force equal to the weight of 16 lbs.; if a force equal to the weight of 10 lbs. just balances the resistance due to friction, after what time will the body have a speed of 80 f.-s.?

13. A projectile striking with a velocity of 1260 f.-s. just pierces an armour plate 14 inches thick. If it pass through a plate 7 inches thick, with what velocity does it emerge, the resistance being uniform?

14. A train of 150 tons acquires in a furlong a velocity of 15 miles an hour; if the resistance due to friction be equal to the weight of 10 lbs. per ton, find the pull exerted by the engine.

15. A projectile whose mass is 700 lbs. has a muzzle velocity of 1600 f.-s. Find the force (supposed to be uniform) which will produce this velocity, if the projectile traverses the bore in $\frac{1}{40}$ second.

16. A mass of 3 lbs. is falling with a velocity of 100 f.-s.; what force will stop it (1) in two seconds, (2) in 2 feet?

IMPULSIVE FORCES.

135. In Art. 134 (1) we saw that the equation $mv-mu=Ft$ made it clear that the *momentum gained* measures the amount of force expended in any time t, and hence the *change* of momentum will in general be very small when the time is very small.

If, however, F be very large, an appreciable change of momentum may be produced in a very short time.

A force acting under such conditions is usually called an **Impulsive Force**.

DEF.—*An* **Impulsive Force** *is a force which produces a definite change of Momentum in a time too short to be measured.*

136. If a large force act for a time so short that it is impossible to estimate it, we cannot measure the force in the ordinary way, and such a force must therefore be measured, not by the *rate of change* of momentum (Art. 124), but by the *total change* of momentum.

The following are examples of Impulsive Forces:—A billiard ball struck with a cue; a cricket ball struck with a bat; a projectile fired from a gun; a nail driven with a hammer; an arrow shot from a bow.

An Impulsive Force (or Impulse) being measured by the Momentum produced by it, if the measure of an Impulse be denoted by I, then

$$mv = I.$$

Where v denotes the total change of velocity.

The effect of an Impulsive Force in producing Momentum is the same as if a much smaller force acted for a finite time.

If we wish to compare an Impulsive and a Finite Force we must know the *time* during which the latter has acted.

DEF.—The *unit of Impulse* is the Impulse which generates the Unit of Momentum.

Example i.—If a ball whose mass is 3 lbs. be started with a velocity of 40 f.-s., find the measure of the blow.

$$I = mv = 3 \times 40 = 120 \text{ units of impulse.}$$

NOTE.—A poundal must act on this body for *two minutes* to produce the same change of momentum.

If a poundal act for this time, the body will move 2400 feet, whereas the blow produced the velocity of 40 f.-s. in an indefinitely short distance.

Example ii.—If a mass of 50 lbs. be fired vertically from a gun with a certain charge, and rise for 20 seconds, through what height will a 40 lb. mass ascend if fired with a double charge?

$$\text{We have } v=u+at;$$
$$\therefore 0=u-32\times 20;$$
$$\therefore \text{initial vel.}=640 \text{ f.-s.}$$
$$mv=I;$$
$$\therefore 50\times 640=I.$$
$$\text{Also } 40\,v=2I;$$
$$\therefore v=\frac{2\times 50\times 640}{40}=1600;$$
$$\text{and } v^2=u^2+2as;$$
$$\therefore 0=(1600)^2-64s;$$
$$\therefore s=\frac{(1600)^2}{64}=40{,}000 \text{ feet.}$$

EXAMPLES—XXIII.

1. A mass starts with a velocity of 100 f.-s.; with what velocity will a mass half as great start under the action of an impulse twice as great?

2. If a ball be thrown up and attain a height of 200 feet, how high will it ascend if started with three times the impulse?

3. Two masses, 3 lbs. and 5 lbs., rise to heights of 300 feet and 200 feet; compare the impulses.

4. If two masses, m and $5m$, are projected vertically upwards with impulses $3F$ and $7F$, compare the heights attained.

5. If two impulses 15 and 35 impart to two masses velocities of 49 and 90 respectively, compare the masses.

6. If a mass under a given impulse rise to a height of 300 feet, and another projected vertically with eight times the impulse be 12 seconds in the air, compare the masses.

7. Two bolts, masses 5 oz. and 8 oz., are fired from a cross-bow; compare the velocities with which they start.

8. A shell lying on the ground is separated by explosion into two fragments whose masses are 100 lbs. and 50 lbs. respectively. If the larger fragment have an initial velocity of 200 f.-s., find the other's velocity.

9. A ball of mass ¼ moving with a velocity of 50 is struck, as it moves, a blow in a direction at right angles to that of its motion, and is by it made to proceed in a direction making an angle of 45° with its original path; find the momentum applied by the blow.

10. A particle of unit mass moving with no acceleration is made by impulses acting instantaneously upon it, when at the several vertices, to describe the perimeter of a regular hexagon, always with the unit velocity; find the momentum applied by each impulse.

11. A billiard ball, struck with a force *F*, describes the length of the table (10 feet) with uniform velocity in 2 seconds; what impulse will cause it to describe the breadth (6 feet) in $1\frac{1}{2}$ seconds?

12. A ball whose mass is 6 lbs. strikes a target with a velocity of 64 f.-s. and stops dead; find the measure of the blow.

13. A ball whose mass is 10 lbs. strikes a wall with a velocity of 80 f.-s., and rebounds along the same line with a velocity of 16 f.-s.; find the measure of the impulse.

14. If a mass of 10 lbs. fall from a height of 36 feet on a fixed horizontal plane and leave it with a velocity of 30 f.-s., find the measure of the blow. If the bodies were in contact for $\frac{1}{32}$ of a second, find the mean pressure between them.

15. A projectile whose mass is m lbs. leaves a gun with a muzzle velocity of 1600 f.-s., and, striking a target with a velocity of 1400 f.-s., is brought to a standstill in $\frac{1}{6}$ second; find the average resistance offered by the target.

16. A projectile whose mass is 1800 lbs. strikes a target with a velocity of 1400 f.-s., and rebounds with a velocity of 120 f.-s.; find the measure of blow delivered.

CHANGE OF UNITS.

137. The Units of Mass, Force, Space, and Time are so connected that if any three of these be given the fourth may be found. The young student is advised to work all questions of this nature as in the following examples, as he can thus, perhaps, best trace and realise what happens when any Unit is changed.

Example i.—If the weight of a pound be selected as the unit of force, one pound of matter as the unit of mass, one second as the unit of time, what is the unit of length? Let x feet = Unit of length.

By definition of Unit Force (Art. 114),

1 lb. weight produces in 1 lb. an accel. of x feet per 1 sec. per 1 sec

But 1 lb. weight ,, 1 lb. ,, g ,,

$\therefore x = g$; $\therefore$ Unit of length $= g$ feet.

Example ii.—If 4 lbs. be selected as unit of mass, a yard as unit of length, 10 secs. as unit of time, what is the unit of force?

Let x poundals = Unit of force;

By definition of Unit Force,

x pdls. produce in 4 lbs. an accel. of 1 yd. per 10 sec. per 10 sec.

$\therefore \frac{x}{4}$ " " 1 lb. " " " "

$\therefore \frac{x}{12}$ " " 1 lb. " 1 ft. " "

$\therefore \frac{10x}{12}$ " " 1 lb. " 1 ft. per 1 sec. "

$\therefore \frac{100x}{12}$ " " 1 lb. " 1 ft. per 1 sec. per 1 sec.

But 1 " " 1 lb. " 1 ft. per 1 sec. per 1 sec.

$$\therefore \frac{100x}{12} = 1; \quad \therefore x = \frac{3}{25};$$

$$\therefore \text{Unit of force} = \frac{3}{25} \text{ poundal.}$$

Example iii.—The unit of force is the weight of 7 lbs., the unit of mass is 21 lbs., the unit of space is 6 feet; what is the unit of time?

Let x seconds = Unit of time.

By definition of Unit Force,

7 lbs. weight produces in 21 lbs. an accel. of 6 ft. per x sec. per x sec.

$\therefore \frac{1}{}$ " " 1 lb. " " " "

$\therefore \frac{1}{18}$ " " 1 lb. " 1 ft. " "

$\therefore \frac{x}{18}$ " " 1 lb. " 1 ft. per 1 sec. "

$\therefore \frac{x^2}{18}$ " " 1 lb. " 1 ft. per 1 sec. per 1 sec.

But $\frac{1}{g}$ " " 1 lb. " 1 ft. per 1 sec. per 1 sec.

$$\therefore \frac{x^2}{18} = \frac{1}{32}; \quad \therefore x^2 = \frac{9}{16}; \quad \therefore x = \frac{3}{4};$$

$$\therefore \text{Unit of time} = \frac{3}{4} \text{ second.}$$

EXAMPLES—XXIII. (2.)

1. If the weight of 2 lbs. be taken as unit force, 4 lbs. as unit mass, 3 yards as unit of length, find unit of time.

2. If a cwt. of matter be the unit of mass, 8 feet the unit of length, 4 seconds the unit of time, find the unit of force.

3. The unit of force is the weight of 10 lbs., the unit of mass is 6400 lbs.; find the unit of length when the unit of time is 10 seconds.

4. If the unit of mass be 10 lbs., the unit of time a minute, the unit of length a fathom, find the unit of force.

5. If unit of force be the weight of a pound at a place where $g=32{\cdot}2$, the unit of mass $16\frac{1}{10}$ lbs., find the unit of time, the unit of length being 8 feet.

6. If the unit of acceleration be 10 f.-s.-s., find unit of mass, the unit of force being the weight of 50 lbs.

7. If the unit of force be the weight of 16 lbs., the unit of mass 1120 lbs., the unit of velocity $9\frac{1}{7}$ f.-s., find the units of length and time, the value of g being 32.

8. If a mile be the unit of length, and a minute the unit of time, compare the measures of the weight and the mass of a body.

9. If in the equation $F=kma$ (Art. 111), we select the following units, Unit Force = weight of 3 lbs., Unit Mass = 8 lbs., Unit Time = 2 seconds, Unit Length = 7 feet, what will be the value of k?

10. Using the units of Ex. 9, find the space passed over in 8 seconds, if 11 lbs. weight act on a mass of 20 lbs. Verify your result by using British Units.

11. Using the units of Ex. 9, find the velocity after 21 seconds if 10 lbs. weight act on a mass of 12 lbs. Verify your result by using British Units.

12. If in the equation $F=kma$, the Unit Force = $2\frac{1}{2}$ lbs. weight, Unit Mass = 112 lbs., Unit Length = $\frac{2}{3}$ foot, and $k=\frac{56}{1215}$, what Unit of Time has been chosen?

CHAPTER V.

THE THIRD LAW OF MOTION.

138. THE First Law of Motion affords a definition of force; the Second Law tells us how to measure force; the Third Law, which will now be given, points out the nature of the force acting.

Third Law.—'*To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.*'

139. *Action* is the name given to the force exerted by one body on another, and *Reaction* is then the force exerted by the latter upon the former.

We may illustrate the force called in one aspect *action*, and in another *reaction*, by any commercial transaction. From one point of view we have *selling*, from the opposite point of view we have *buying*, and when both sides are considered we call the transaction *trade*. To pursue the simile,—as an auditor, in examining books, must bear in mind in whose interest the respective sides are made up, so we must always remember which of the two bodies we are concerned with, in order that we may write down the forces in the interest of that body, and not set down any force on the wrong side of the account.

140. The term *Stress* is often used to denote the *whole* of the mutual action between two bodies. The stress is measured numerically by the force exerted on *either* body.

The scientific use of this word is due to Professor Rankine.

Pressure is the stress when two bodies are in contact, and the force exerted *on* either is *directed away* from the other.

Thus, when a mass rests on a horizontal table, the force exerted on the table is downwards, the equal force exerted on the body is upwards. If we denote this latter force by R, then $R=mg$.

Tension is the stress when two bodies are connected by a string, and the force exerted *on* either is *directed towards* the other.

Thus, when a mass is suspended by a string from a fixed support, the force which keeps the body in its place is directed upwards, the force exerted on the point of support is directed downwards. If we denote the former force by T, then $T=mg$.

Attraction is the stress when bodies act on each other at a distance without our perceiving how the force is exerted, and the mutual action tends to bring the bodies together.

Repulsion is the stress under similar circumstances, and the mutual action tends to separate the bodies.

141. The following **Illustrations** of the Third Law may be given:—

1. If any body press against another, it is pressed by this other with an equal force in the opposite direction; *e.g.*, a mass resting on a table; a finger pressed against a stone.

2. If any body draw another, it is drawn by this other with an equal force in the opposite direction; *e.g.* when a horse tows a boat on a canal, the horse is drawn backwards by a force equal to that which he impresses on the towing-rope forwards.

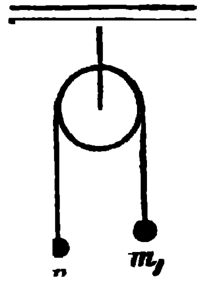
3. By whatever amount, and in whatever direction, one body has its momentum changed by impact with another, this other body has its momentum changed by the same amount in the opposite direction; *e.g.* when one billiard ball strikes another, at each instant during impact the force between them is equal and opposite on the two bodies, and will therefore produce the same change of momentum in each. (*See* Art. 103.)

4. When one body attracts or repels another, this other attracts it with an equal and opposite force; *e.g.* when a stone falls from a height, the earth ascends to meet it, and the force acting on the stone is equal and opposite to the force acting on the earth.

This, of course, cannot be made the subject of direct experiment, but we know that the Third Law is found to be true wherever it can be tested, and the statement is a legitimate deduction from this Law.

142. *When two unequal masses are connected by a string passing over a smooth pulley; to determine* (1) *the Acceleration;* (2) *the Tension of the string.*

Let m and m_1 be the masses of the two bodies ($m > m_1$), and T be the tension of the string, which is the same throughout.



Consider m. Then using the Kinetic measure of force, the forces acting on m are its *Weight* (mg) *downwards*, and the *Tension* (T) *upwards*. And since m is descending, the resultant force acting on m is therefore $mg - T$ downwards.

If a be the acceleration produced by this force, then

$$ma = mg - T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.)$$

Similarly, the force acting on m_1 is $T - m_1g$ upwards.

$$\therefore \; m_1a = T - m_1g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.)$$

To find a, we add these equations;

$$\therefore \; (m + m_1)a = (m - m_1)g;$$

$$\therefore \; a = \frac{m - m_1}{m + m_1} \cdot g.$$

To find T, divide equation (1) by (2).

$$\therefore \; \frac{m}{m_1} = \frac{mg - T}{T - m_1g};$$

$$\therefore \; mT - mm_1g = mm_1g - m_1T;$$

$$\therefore \; (m + m_1)T = 2mm_1g;$$

$$\therefore \; T = \frac{2mm_1}{m + m_1} \cdot g.$$

The student is advised not to use these formulæ in working examples, but to deduce the required results each time by applying the Dynamical Equation of Article 124.

Having found the measure of the acceleration, he can then apply the formulæ of Article 64, to find the velocity acquired, the space described, and allied questions.

Example i.—Two masses, of 9 lbs. and 7 lbs., are connected by a string passing over a pulley; find the acceleration and the tension.

For the 9 *lbs.*, $9a=9g-T.$

For the 7 *lbs.*, $7a=T-7g.$

To find a, $16a=2g$; $\therefore$ $a=4$ f.-s.-s.

To find T, $\frac{9}{7}=\frac{9g-T}{T-7g}.$

From which $T=252$ poundals; or, $T=7\frac{7}{8}$ lbs. weight.

Example ii.—Two weights, 17 and 15 pounds respectively, are connected by a string passing over a pulley; how long will the heavier take to descend 144 feet? (We use the Gravitation measure of force).

The *masses* are $\frac{17}{g}$ and $\frac{15}{g}$ respectively.

For the 17 *lbs.*, $\frac{17}{g}.\,a=17-T.$

For the 15 *lbs.*, $\frac{15}{g}.\,a=T-15.$

To find a, $\frac{32}{g}$, $a=2$ f.-s.-s.

Now $s=ut+\frac{1}{2}at^2$;

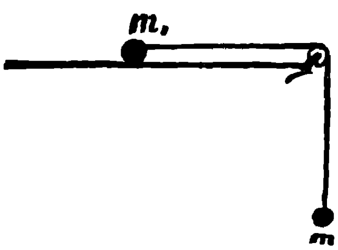
$\therefore$ $144=0+\frac{1}{2}\times 2\times t^2$;

$\therefore$ $t=12$ seconds.

See Examples XXIV. Nos. 1-8.

143. *When a mass hanging freely draws another mass along a smooth horizontal table by means of a string passing over a pulley at its edge; to find* (1) *the Acceleration*; (2) *the Tension.*

The weight of m_1 acting vertically is balanced by the reaction of the table, and therefore will not affect the horizontal motion of the body, which is caused by the Tension alone.



Consider m, $ma=mg-T$ . . . (1.)

Consider m_1, $m_1a=T$ . . . . . (2.)

To find a, $(m+m_1)\,a=mg$;

$$\therefore\ a=\frac{m}{m+m_1}.g;$$

To find T, $\frac{m}{m_1}=\frac{mg-T}{T}$;

$$\therefore\ mT=mm_1g-m_1T;$$

$$\therefore\ (m+m_1)\ T=mm_1g;$$

$$\therefore\ T=\frac{mm_1}{m+m_1}\,.\,g.$$

Example.—If a mass of 3 lbs. hanging freely draw a mass of 13 lbs. along a smooth table, find the space described in 5 seconds from rest.

For the 3 *lbs.,* $3a=3g-T.$

For the 13 *lbs.,* $13a=T.$

To find a, $16a=3g;$

$\therefore\ a=6$ f.-s.-s.

To find the space described, $s=ut+\frac{1}{2}a.t^2;$

$\therefore\ s=0+\frac{1}{2}\times 6\times 25$

$=75$ feet.

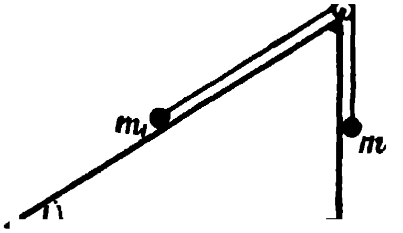
See Examples XXIV. Nos. 9-16.

144. *When a mass hanging freely draws another mass up a plane by a string passing over a pulley at the vertex; to determine* (1) *the Acceleration;* (2) *the Tension of the string.*

For m, $ma=mg-T$ . . . . (1.)

For m_1, $m_1a=T-m_1g\sin i$. (2.)

(*See* Art. 133.)



To find a, $(m+m_1)a=(m-m_1\sin i)g.$

$$\therefore\ a=\frac{m-m_1\sin i}{m+m_1}\,.\,g.$$

To find T, $\frac{m}{m_1}=\frac{mg-T}{T-m_1g\sin i}.$

From which equation $T=\frac{mm_1(1+\sin i)}{m+m_1}\,.\,g.$

Example.—A mass of 9 lbs. hanging freely draws a mass of 7 lbs. up a plane whose inclination is 30°. Motion continues for 4 seconds, and then the string breaks; how far will the 7 lbs. move up the plane after that?

For the 9 *lbs.*, $9a=9g-T.$

For the 7 *lbs.*, $7a=T-7g\sin 30°.$

$$\therefore\ 16a=\left(9-\frac{7}{2}\right)32;$$

$$\therefore\ a=11 \text{ f.-s.-s.}$$

To find velocity acquired during the motion—

$$\begin{aligned} v&=u+at\\ &=0+11\times 4\\ &=44 \text{ f.-s.}\end{aligned}$$

When the string breaks, this velocity at once becomes subject to a retardation of $g\sin 30°=16$ f.-s.-s.

$$v^2=u^2+2as;$$

$$\therefore\ 0=44^2-2\times 16\times s.$$

From which equation we get $s=60\frac{1}{2}$ feet.

See Examples XXIV. Nos. 17-27.

145. In a similar way, if *two masses* m *and* m_1 *are connected by a string passing over the common vertex of two inclined planes whose inclinations are* i *and* i_1 *respectively*, it may be shown that

$$a=\frac{m\sin i-m_1\sin i_1}{m+m_1}.g.$$

$$\text{and,}\quad T=\frac{mm_1(\sin i+\sin i_1)}{m+m_1}.g.$$

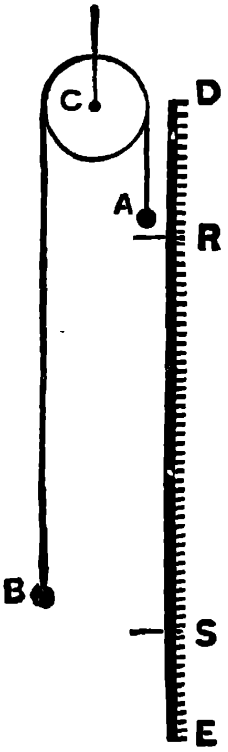
The student may compare these results with those in Article 142, and he will notice that either of these problems may be regarded as a special case of the other, it being understood that when a body hangs freely it may be conceived as resting on a plane whose inclination is 90°.

See Examples XXIV. Nos. 28-30.

ATWOOD'S MACHINE.

146. An accurate knowledge of the numerical value of g at any place is of great importance. The most trustworthy results are obtained by the number of oscillations made by a pendulum in a given time, but fairly satisfactory results can be obtained by means of the apparatus now to be described. These results may be depended on, within reasonable limits, in the machines furnished with modern refinements of construction and means of measuring extremely small intervals of time.

The essential object secured by Atwood's machine is, that by its use we can so modify the force of gravity that its effect can be accurately measured.



147. Two bodies A and B of equal mass (m) are connected by a very fine silk string which passes over a small pulley C. The axis of this pulley rests on wheels which greatly diminish friction, and known as *friction wheels*.

The machine is furnished with (1) a pendulum beating seconds; (2) a vertical scale DE, at any point of which a ring R (through which A can pass) and a platform S may be clamped at pleasure.

The masses of A and B being equal, no motion takes place; but if a small bar (mass $=x$) be placed on A, the weight of the bar will cause A to descend, and, of course, B to ascend with a certain acceleration.

Let A, with the bar resting on it, be raised to the level D, and, by an automatic action as the pendulum begins to oscillate, let A begin to move. If now R be clamped to the scale so that A may reach it in one second, it follows that the weight of the bar has acted on the mass $2m+x$ for a second. A passes through the ring, but the bar is caught. The masses in motion now being equal, and therefore equal forces acting on A and B, each will move uniformly with the velocity which it had at the instant the bar was detached. (By the First Law.) If therefore S be fixed to the scale at such a point that A reaches it in one second after passing R, the space RS will measure this velocity, and therefore *the velocity at the end of the first second*, and this measures the acceleration produced by the weight of the bar in the mass $2m+x$. (Art. 51.)

It is found that if x be small and $2m$ be considerable, we can make this velocity so small as to be easily measured.

148. By this machine some of the results already arrived at can be put to the test of experiment.

Experiment i.—Let A and B be composed of *equal discs*, made so that they can be removed, or added to, at pleasure, and be arranged so that any number can be stopped at will on passing through R.

If the number composing A and B be equal, there will be no motion. Let the total mass be $2m$.

If we transfer one or more discs from B to A, we do not change the mass moved, but we do change the force which causes the motion. If the space RS be measured after each transfer has been made, as the result of such experiments it is found that—

> *When the mass moved is the same, the acceleration is always proportional to the force acting.* (*See* Art. 109.)

Experiment ii.—If a certain number of discs, n suppose, be placed on A, and RS be measured, we have a certain acceleration produced by the weight of n discs: if now we *add* the *same* number of discs to A and B, or *take* the *same* number from A and B, we have still the same force (=weight of n discs) producing motion, but we have changed the mass moved; and if RS be measured in each case, it is found as the result of such experiments that—

> *When the force remains the same, the acceleration is inversely proportional to the mass moved.* (*See* Art. 111.)

149. It has been shown in Article 142, that when any two masses $(m+x)$ and m are arranged as in this machine,

$$\text{then, } a=\frac{x}{2m+x}.g; \qquad \therefore\ g=\frac{2m+x}{x}.a.$$

Now the value of a is obtained from the scale, being the value of *RS*, and the values of $2m$ and x being also known, the numerical value of g at the place may be computed from this equation. *See* Examples XXIV. Nos. 67-69.

EXAMPLES—XXIV.

1. A mass of 7 lbs. is connected by a string passing over a smooth pulley with a mass of 4 lbs. ; find the space described in 5 seconds.

2. A mass of 7 is connected with a mass of 5, as in the last example; find the space described in 10 seconds.

3. Masses of 9 and 7 are connected by a string over a pulley; find the space described in 8 seconds, and in the 8th second.

4. Masses of 8 and 5 are connected by a string over a pulley ; find the space in 5 seconds, and the velocity after 10 seconds.

5. The masses are 5 and 4; after the greater has descended 4 feet, the string breaks; how far will each move in the next second?

6. The masses are 3 and 5 lbs. ; after 1 second the string breaks; for how long, and how far, will the 3 lbs. mass ascend?

7. The masses are 5 and 7 lbs. ; after 2 seconds the string breaks; find the velocity of the 5 lbs. mass when it passes through its starting-point.

8. A mass of $17\frac{1}{2}$ lbs. draws one of $14\frac{1}{2}$ lbs. over a fixed pulley; find the velocity of each after describing 2 feet from rest.

9. A mass of 10 lbs. hanging freely draws a mass of 55 lbs. along a smooth table; find the acceleration, the space described in 5 seconds; the space in 8th second, and the velocity acquired between the 7th and 12th second.

10. Find the acceleration when a mass of 1 lb. hanging freely draws a mass of 1 cwt. along a smooth horizontal table.

11. In what time will a mass of 1 lb. hanging freely draw a mass of 10 lbs. along a smooth horizontal table, the length of which is 20 feet?

12. A mass of 13 lbs. lying on a table is connected by a string over the edge with a mass of 3 lbs. hanging freely; find the space described by each body in 3 seconds.

13. If the mass hanging freely, in the last question, be 5 lbs., find the space described in 2 seconds.

14. A mass of 5 lbs. hanging freely draws 15 lbs. along a smooth horizontal table. When motion begins the 15 lbs. mass is 12 feet from the edge; when will it reach the edge?

15. A mass of 25 lbs. hanging vertically draws a mass of 100 lbs. along a smooth table; find the tension in the string.

16. A mass of 12 oz. is drawn along a table by a mass of 4 oz. hanging vertically; find the velocity after 4 seconds; and if the string then breaks, find the space described by each body in 6 seconds more.

17. A mass of 20 lbs. hanging freely draws a mass of 16 lbs. up a plane whose inclination is 30°; find the acceleration, and the tension in the string.

18. A mass of 5 lbs. hanging freely draws a mass of 6 lbs. up the plane in the last example; find the space described in 11 seconds.

19. A mass of 25 lbs. hanging freely draws a mass of 45 lbs. up a plane 450 feet long, whose inclination is 30°. Where should the string be cut so that the 45 lbs. may just reach the top?

20. If a mass of 29 lbs. draws a mass of 28 up a plane whose inclination is 30°, find the space described in 10th second from rest.

21. A waggon, whose mass is $\frac{1}{2}$ ton and loaded with a mass of 1 ton, moves down a plane whose inclination is 30° and length 576 feet, dragging a mass of $\frac{1}{2}$ ton by a string over a pulley at the vertex; with what velocity does the waggon reach the bottom of the plane?

22. A mass of 5 lbs. hanging freely draws a mass of 3 lbs. up a plane whose inclination is 30°. After 1 second the string breaks; how far will the 3 lbs. mass ascend after that?

23. A mass of 1 lb. hanging freely is connected with a mass of 7 lbs. resting on a plane whose inclination is 30°, and length 92 feet. The 7 lbs. mass is placed close to the pulley. After 1 second the string parts; in what time after that will the 7 lbs. mass reach the bottom?

24. A mass of 13 lbs. hanging freely draws a mass of 8 lbs. up a smooth inclined plane rising 3 in 4; find the acceleration.

25. A mass of m lbs. hanging freely draws a mass of 10 lbs. up a smooth inclined plane whose inclination is 45°; find m if the acceleration be $\frac{1}{2}g$.

26. A mass of 15 lbs. hanging freely draws a mass of 9 lbs. up an inclined plane. If 81 feet be described in 3 seconds from rest, find the inclination of the plane.

27. A mass m on a smooth inclined plane is connected by a string over a pulley with a mass $\frac{3}{4}m$ hanging freely; find the inclination when m moves up a distance $\frac{1}{8}g$ in the first second.

28. Masses of 3 and 4 rest on two planes whose inclinations are 45° and 60°, and are connected by a string over the common vertex; find the acceleration.

29. If the masses in the last example are 12 and 4 lbs., find the acceleration, the velocity acquired in 4 seconds, and the space described in 6 seconds.

30. If masses of 8 and 10 rest on two planes whose inclinations are 60° and 30°, find the tension in the string, and the space described in the 7th second.

31. Two masses of 10 lbs. each are connected by a string over a pulley; what mass must be added to one of them to cause it to descend 100 feet in 4 seconds?

32. If the masses m and m_1 are connected by a string over a smooth peg, find the ratio of m to m_1 if the acceleration is $\frac{1}{4}g$.

33. In the last example find the ratio of m to m_1 if m descends 32 feet in 2 seconds.

34. If m and 181 are connected by a string over a pulley, find m if 181 descend 18 feet in 3 seconds.

35. If two masses (each = 93 lbs.) be connected by a string over a pulley, what mass must be added to one of them that it may descend 100 feet in 8 seconds?

36. If the masses be 50 lbs. each, what mass must be added to one of them to cause that mass to descend 32 feet in 4 seconds?

37. Masses of 3 lbs. and 6 lbs. are connected with a mass of 7 lbs. by a string over a pulley. After 4 seconds the 3 lbs. mass is detached; how long, and how far, will the 6 lbs. mass descend?

38. Masses of 7 and 5 are connected with a mass of 9. After 14 seconds the 7 is detached; through what space will the 9 ascend?

39. Two masses of 9 lbs. each are connected by a string over a pulley; what mass must be added to one of them that it may descend 100 feet in 5 seconds?

40. A mass m is drawn up a plane whose inclination is 30° by a mass m_1 hanging freely; if the acceleration is $\frac{1}{4}g$, find the ratio of m to m_1.

41. If in last question the acceleration is $\frac{1}{2}g$, find ratio of m to m_1.

42. A string is just strong enough to support a tension equal to $\frac{1}{3}$ the sum of the weights of the masses at the extremities when the string is placed over a pulley; find the least possible acceleration that the string may not break.

43. Prove that when two masses are suspended by a string over a smooth pulley, the tension is less than half the sum of the weights.

44. If in Atwood's machine the total pressure on the points of support be 6 lbs. weight, and the sum of the weights of m and m_1 be 16 lbs. weight, find m and m_1.

45. Two equal masses are connected by a string over a pulley; one of them is started with a velocity v; show that both will retain this velocity, and find the tension of the string during motion.

46. A mass of 1 lb. is raised by a mass of 2 lbs. connected with it by a string over a pulley. If the string break when the 1 lb. has been raised through $\frac{3}{4}$ height of the pulley above the ground, show that it will just reach the pulley before it begins to descend.

47. If in Atwood's machine one of the masses be 10 lbs., and the string can only bear a tension of 12 lbs. weight, find the mass at the other end of the string.

48. Two masses m and m_1 hang over a pulley; m descends 18 feet in 3 seconds. If 4 oz. had been added to m_1, then m_1 would have descended 16 feet in $4\frac{1}{2}$ seconds; find m and m_1.

49. Two masses of 10 lbs. and 2 lbs. are connected by a string over a pulley at the top of an inclined plane whose length is 112 feet and inclination is 30°. The greater rests on the plane close to the pulley, the less hangs freely. The string breaks 2 seconds after motion has commenced. In what time will the 10 lbs. mass reach the bottom?

50. If m, the greater of two masses connected by a string over a pulley, descend with an acceleration $= a$, show that the mass which must be taken from it in order that it may *ascend* with the same acceleration $= \frac{4ga}{(g+a)^2} \cdot m$.

51. Two masses m and m_1 are connected as in Atwood's machine; if the pulley can bear a strain of half the sum of the weights of the masses, find the least possible ratio of m to m_1.

52. If two equal masses hang over a pulley, and one be projected up with a velocity of $\frac{1}{2}g$, in what time will the string become taut?

53. A number of equal masses are fastened to different parts of a string, which is then placed over a smooth pulley; show that at any time the tensions of the successive parts of the string are, on each side, in A. P.

54. A uniform string hangs at rest over a peg. From one end one-fourth of the whole length of the string is cut off; show that the pressure on the peg is at once reduced by one-third of the weight of the whole string.

55. Two equal masses are connected by a taut string l feet long, one hanging over the edge of a table and the string at right angles to the edge; with what velocity does the other mass leave the table, supposing the height of the table to be x feet?

56. A mass of twelve pounds is so distributed at the extremities of a string passing over a pulley that the greater descends through 7 feet in as many seconds; find the mass at each end of the string

57. Describe Atwood's machine, and explain how it may be employed to discover the numerical value of the acceleration due to gravity at any place.

58. One of two equal particles connected by a string rests on a perfectly smooth horizontal table, the other hanging over the edge; show that the motion is the same as if either particle moved freely down a plane inclined to the horizon at an angle of 30°.

59. A mass of 10 lbs. is attached to one end of a string passing over a smooth pulley; find the mass which must be attached to the other in order that the acceleration may be half that of gravity.

60. If in an Atwood's machine the string can bear a tension equal to only $\frac{4}{9}$ of the sum of the weights, show that the least acceleration possible is $\frac{1}{3}g$.

61. Two masses of 9 oz. and 7 oz., connected by a thread 36 feet long, are hung over a small fixed pulley so that the string is stretched and the mass of 7 oz. at first touches the ground. After motion has lasted for 2 seconds, the thread is cut, and it is found that both bodies reach the ground at the same time; what is the height of the pulley from the ground?

62. In Atwood's machine equal masses of 10 oz. are suspended by a string which passes over the pulley, and a bar of 1 oz. is placed across one of them. This, after passing through the space of 1 foot, passes through a ring which removes the 1 oz.; how far will the body descend in the next minute?

63. A string passes over a smooth pulley, and has attached at one end a mass of 8 lbs., and at the other two masses of 6 and 4 lbs. After running down for 2 seconds, the mass of 6 lbs. is gently removed; how much further will the 8 lbs. ascend?

64. A mass of 5 lbs. is placed on a smooth inclined plane of slope 1 in 2. A mass of 1 lb. is attached to this by means of a cord, which passes over a pulley at the top of the plane and hangs freely; investigate the motion.

65. Two masses of 6 lbs. and 8 lbs., 2⅔ feet apart, together pull a mass of 10 lbs. over a smooth pulley by means of a string. After falling 4 feet the 8 lbs. mass reaches ground; show that the 6 lbs. mass will just reach the ground.

66. Two bodies of weight *W* and 3*W* respectively are connected by a string which passes over a smooth fixed pulley; show that the acceleration of the C. G. is one-fourth of the acceleration of a body falling freely.

67. The masses in Atwood's machine are 4 oz. each, and a bar whose mass is ¾ oz. is placed on one of them. After motion has continued for a second the bar is gently removed. The space traversed in the next second by the descending body is 2·76 feet; find the local value of *g*.

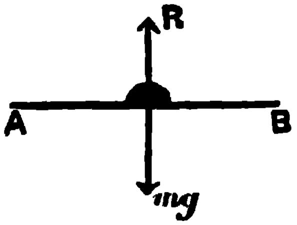
68. The masses at the ends of the string are 8½ oz. each, the mass of the bar is ⅔ oz.; after motion has taken place for a second the bar is removed, and the space passed over in the following second is 1·214 feet; find from this experiment the local value of *g* correct to two places of decimals.

69. The masses at the ends of the string are 15¾ oz. each and the bar is ½ oz.; find the value of *g*, when, after the bar has rested for a second only on one of the bodies, that body descends 1 foot in the next 2 seconds.

PRESSURE ON A PLANE IN MOTION.

150. We may now consider the Pressure exerted by a mass resting on a horizontal plane which has vertical motion.

151. Let us first suppose that the plane *AB* is *descending* with uniform acceleration $= a$. Let a body, whose mass $= m$, rest on *AB*. Then, by the Third Law, the Pressure on *AB* = Pressure on body.



Let this Pressure $= R$.

Since m is moving with an acceleration a,

$\therefore ma =$ the force causing m to move.

This force is the resultant of all the forces acting on m, *i.e. of the Weight of m downwards*, and *the Pressure of the plane against the body upwards.*

Since the body is *descending*, the Weight exceeds the Pressure,

$\therefore$ $mg-R$ is the force causing m to descend;

$$\therefore ma=mg-R;$$
$$\therefore R=mg-ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(1.)}$$

152. Next, let us suppose that the plane AB is *ascending* with uniform acceleration $=a$.

Then, by similar reasoning, the pressure upwards exceeds the Weight downwards.

$$\therefore ma=R-mg;$$
$$\therefore R=mg+ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(2.)}$$

153. *Special Case.*—If the platform is ascending or descending with *uniform velocity*, then $a=0$, and from each of the equations (1) and (2), we have—

$$R=mg.$$

We infer, therefore, that the Pressure exerted is the *Weight of m*, and therefore the same as if the plane were at rest.

Example i.—A ball whose mass is 12 lbs. is held on the hand, and made to descend with an acceleration of 16 f.-s.-s.; find the pressure exerted on the hand.

Let R = pressure exerted on ball by the hand.

$$\begin{aligned}\therefore ma&=mg-R\\ \therefore R&=mg-ma\\ &=m(g-a)\\ &=12\,(32-16)\\ &=12\times16 \text{ poundals}\\ &=6 \text{ lbs. weight.}\end{aligned}$$

Example ii.—A balloon is ascending vertically, and a pound mass presses with a force equal to the weight of 17 oz. on the aëronaut's hand; find the height attained by the balloon in the first 20 seconds.

Let a = acceleration of the balloon.

$$ma = R - mg$$

$$\therefore\ 1 \cdot a = \frac{17}{16}g - 1 \cdot g$$

$$\therefore\quad a = \frac{1}{16}g = 2 \text{ f.-s.-s.}$$

Then $s = ut + \frac{1}{2}at^2$

$$\therefore\quad s = 0 + \frac{1}{2} \times 2 \times 400 = 400 \text{ feet.}$$

Example iii.—If a man jump off a height with a mass of 56 lbs. resting on his head, what pressure is exerted on his head while in the air?

Let R = pressure on his head.

Then the whole system is moving downwards with an acceleration $= g$;

$\therefore\ ma$ = weight down − pressure up;

$$\therefore\ 56 \cdot g = 56 \cdot g - R;$$

$$\therefore\ R = 0;$$

i.e. the mass is just in contact with his head, but exerts no pressure on it.

EXAMPLES—XXV.

1. A balloon carries a mass of 100 lbs. suspended by a rope from the car; find in kinetic measure the tension in the rope:—

(*a*) When the balloon ascends with uniform velocity of 16 f.-s.

(*b*) ,, ,, ,, acceleration of 16 f.-s.-s.

(*c*) ,, descends ,, ,, 16 f.-s.-s.

(*d*) ,, ascends ,, ,, 24 f.-s.-s.

(*e*) ,, descends ,, ,, g.

2. A mass of 140 lbs. rests on a platform; find the pressure on the platform in gravitation measure:—

(*a*) When the platform is stationary.

(*b*) When it is ascending or descending uniformly.

(*c*) When it is ascending with uniform acceleration $\frac{1}{10}g$.

(*d*) ,, descending ,, ,, $\frac{3}{8}g$.

(*e*) ,, descending with a velocity which *increases* every second at the rate of g feet per second.

(*f*) When it is descending with a velocity which *diminishes* every second at the rate of (1) g feet per sec.; (2) 8 feet per second.

3. If I jump off a platform with a mass of 40 lbs. in my hand, find the pressure on my hand while in the air.

4. If a mass of 1000 lbs. exert a pressure on a lift of 750 lbs. weight, what is the acceleration of the lift?

5. A platform supporting a mass of 12 oz. sustains a pressure of $8\frac{1}{4}$ oz. weight; find the acceleration.

6. A ball whose mass is 4 lbs., held in the hand, is (*a*) lowered with a uniform velocity v; (*b*) lowered with uniform acceleration $\frac{1}{2}g$; (*c*) lifted with uniform acceleration $2g$; (*d*) lifted with uniform acceleration g; find the pressure exerted on the hand in each case.

7. Two scale-pans, each of 1 lb. mass, contain masses of 5 lbs. and 3 lbs. They are connected by a string passing over a pulley; find the pressure between the larger mass and the pan.

8. A 1 lb. mass exerts a pressure on the hand of an aëronaut equal to the weight of 22 oz.; find the acceleration of the balloon.

9. In an ascending balloon a mass of 1 lb. produces a downward pressure on the hand of the aeronaut equal to the weight of 18 oz.; find the height to which the balloon has ascended in a minute from the moment of starting.

10. A mass of 15 lbs. hangs by a string from the roof of a railway carriage in rapid motion; find the direction and magnitude of the tension in the string.

11. If the train in the last example went over a precipice, what would be the direction of the string and the tension in it?

12. If the two weights, P and Q, in an Atwood's machine rest in scale-pans of weight W attached to the ends of the string, find the pressures on the two pans.

13. A spring balance is graduated at Greenwich, where $g=32\cdot19$; at Ascension, where $g=32\cdot095$, a body is weighed by the instrument and its mass is apparently 64·19 oz. What is its real mass?

14. A body placed on a spring balance in the car of a balloon at rest appears to weigh 100; what will it appear to weigh if the balloon (1) be rising with an acceleration of 8 f.-s.-s., (2) be descending with an acceleration of 8 f.-s.-s.?

RECOIL.

154. When a cannon is fired, the projectile and the gun move in opposite directions with velocities which depend

on the relative masses of the gun, projectile, and the charge of powder.

The motion is due to the force exerted by the powder gas in one direction on the projectile, in the opposite direction on the gun. If the inertia of the gas be not neglected, the impulse on the gun is somewhat greater than on the shot. We shall suppose the impulses to be exactly equal.

The recoil is checked by an apparatus called the 'Compressor.' This produces a gradually increasing resistance to the motion of the gun, and destroys the velocity of the recoil in a few feet as a rule.

155. *To find the Velocity of the recoil of a gun when a projectile is fired from it.*

Let m = mass of the gun; m_1 = mass of the projectile.
v = velocity ,, v_1 = velocity ,,

Now by the Third Law, Momentum of the gun = Momentum of the projectile, because the force of the explosion produces the same momentum in each.

$$\therefore mv = m_1 v_1.$$

Example i.—If a projectile whose mass is 700 lbs. leave a gun whose mass is 35 tons with a muzzle velocity of 1600 f.-s., find the velocity of the recoil.

Momentum of gun = Momentum of projectile.
Let v = velocity of recoil;

$$\therefore 35 \times 2240 \times v = 700 \times 1600;$$

$$\therefore v = 14\frac{2}{7} \text{ f.-s.}$$

Example ii.—If the action of the compressor be a steady resultant pressure of 10 tons weight, how far will the gun in the last example recoil?

$$ma = F;$$

$$\therefore 35 \times 2240a = 10 \times 2240g;$$

$$\therefore a = \frac{64}{7} \text{ f.-s.-s.}$$

Then $v^2 = u^2 + 2as$;

$$\therefore\ 0 = \left(\frac{100}{7}\right)^2 - \frac{128}{7}\cdot s;$$

$$\therefore\ s = 11\frac{9}{56} \text{ feet};$$

i.e. the gun will recoil $11\frac{9}{56}$ feet.

Examples—XXVI.

1. If a shot whose mass is 20 lbs. leave a gun whose mass is 3 tons with a velocity of 1200 f.-s., find the velocity of recoil.

2. If a shot whose mass is 40 lbs. leave a gun whose mass is 4 tons with a velocity of 1000 f.-s., find the velocity of recoil.

3. A projectile whose mass is 112 lbs. leaves a gun whose mass is 4 tons with a velocity of 900 f.-s.; find the velocity of recoil.

4. A projectile whose mass is ½ ton is discharged from a gun whose mass is 81 tons, with a velocity of 1600 f.-s.; find the velocity of recoil.

5. How far does the gun recoil in the last question, if the compressor exert a steady pressure of 4½ tons weight?

6. If a 64-lb. shot leave a gun whose mass is 95 cwt. with a velocity of 1160 f.-s., how far will the gun recoil if the friction between the gun and the ground be equal to the weight of a ton?

7. If a 600-lb. shot leave a 35-ton gun with a velocity of 1600 f.-s., how far will the gun recoil up a smooth inclined plane, rising 1 in 14?

8. If a projectile of 700 lbs. leave a gun whose mass is 50 tons, and the gun recoil against a steady pressure of 3 tons weight a distance of 4 feet, find the velocity of the projection.

9. A projectile whose mass is 1800 lbs. is fired from a gun whose mass is 110 tons with a velocity of 2140 feet; find the velocity of recoil.

10. What force will bring the gun in Ex. 9 to a standstill in 15 feet?

11. If the projectile (mass 322 lbs.) leave a 12½-ton gun with a velocity of 1400 f.-s., how far will the gun recoil up a smooth inclined plane rising 1 in 4? (Take $g = 32{\cdot}2$.)

12. A projectile whose mass is 800 lbs. is fired from a 43-ton gun; if the muzzle velocity be 1720 feet, how far will the gun recoil up a slope rising 1 in 280, the friction amounting to 20 lbs. weight per ton?

CHAPTER VI.

FUNDAMENTAL PROPOSITIONS IN STATICS.

156. In Articles 131, 132, it was stated that by means of the Second Law of Motion, and certain important Propositions relating to velocities and accelerations, analogous propositions could be established for forces.

These Propositions relating to forces may be here given. As a force is measured by the rate of change of momentum which it can produce, and, when acting on a single body, this rate of change of momentum is measured by the rate of change of velocity—*i.e.* by the acceleration—and acceleration is measured by the velocity produced in a second; so we infer that the straight line which represents the magnitude and direction of a velocity will also represent the magnitude and direction of the force which can produce that velocity in a second.

Thus since a force possesses the three elements of magnitude, direction, and a point of application, and a straight line possesses the same three elements, we infer that a force may be completely represented by a straight line. (Compare Art. 93.)

157. If a straight line drawn from any point O to another point A represent the line in which the force is acting, then if the force tends to cause motion from O to A, we say that OA is the *direction* of the force, and either OA or AO is the *line of action* of the force.

158. DEF.—**A Resultant Force** is the single force which produces the same effect as two or more forces acting on a particle or body. And these several forces which can be thus replaced by the Resultant are known as **Component Forces**, or simply Components.

159. The *Composition of Forces* means the process of finding the resultant when the components are given. The *Resolution of Forces* means the process of finding two or more components when the resultant is given. (Compare Art 16.)

160. If two forces act on a particle, and their lines of action be not in the same straight line, their resultant may be found by the theorem known as the *Parallelogram of Forces.*

THE PARALLELOGRAM OF FORCES.

STATEMENT.—*If two forces, acting on a particle, be represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal which passes through their intersection.*

This important theorem has been proved in Article 130.

If P and Q be the two forces, θ the angle between their directions, and R the resultant, then—

$$R^2=P^2+Q^2+2PQ\cos\theta.$$

This may be proved by Trigonometry, as in Article 19.

161. *Special Cases.*—(1) When the directions of the forces P and Q are at right angles to each other, then $\theta=90^\circ$, and the last formula becomes $R^2=P^2+Q^2$.

(2) When their directions are the same, then $\theta=0^\circ$, and the formula becomes

$$R^2=P^2+Q^2+2PQ;\text{ and }\therefore R=P+Q.$$

(3) When their directions are opposite, then $\theta=180^\circ$, and the formula becomes

$$R^2=P^2+Q^2-2PQ;\text{ and }\therefore R=P-Q.$$

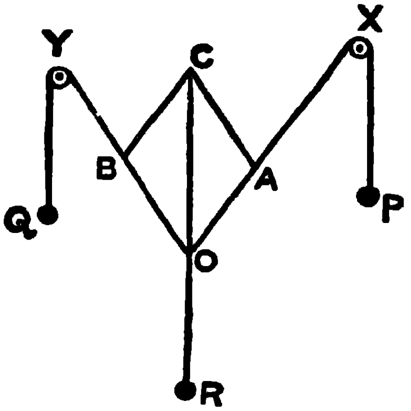
If $P=Q$ in the third case, then $R=0$, *i.e.* two equal forces acting on a particle in opposite directions balance, and therefore keep the particle in equilibrium.

Hence, when two forces act on a particle, their *greatest* resultant is the *sum* of the forces, and their *least* resultant is the *difference* of the forces.

Note.—If the angle between the directions of the two forces increases, the magnitude of the Resultant is diminished.

This may be proved by the methods of Article 19.

162. The following is an *Experimental Proof* of the Parallelogram of Forces :—



Three fine silk strings are knotted at a point O, and three masses, whose weights are P, Q, and R, are tied to their extremities. Two of the strings are placed over smooth pulleys, X and Y, fixed to a vertical board, and the system is then allowed to take up its own position of rest, as in the figure. The point O is evidently acted on by three forces, P and Q acting along the lines OX and OY, and R acting vertically downwards.

If OA and OB be measured along OX and OY respectively, proportional to the weights P and Q, and the parallelogram $OACB$ be completed, it will be found that—

(1) The point C is vertically above O.

(2) The line OC is proportional to the weight R.

Therefore OC, the diagonal of the parallelogram $OACB$, will represent, in direction and in magnitude, the resultant of the forces represented by OA and OB.

TRIANGLE OF FORCES.

163. Statement.—*If a particle be acted on by three forces which are represented in magnitude and direction by the sides of any triangle taken in order, the forces will be in equilibrium.*

This may be proved by the method of Article 20.

CONVERSELY.—*If three forces acting on a particle be in equilibrium, and if any triangle be drawn, having its sides parallel to the directions of these forces, the sides of the triangle will also represent the forces in magnitude.*

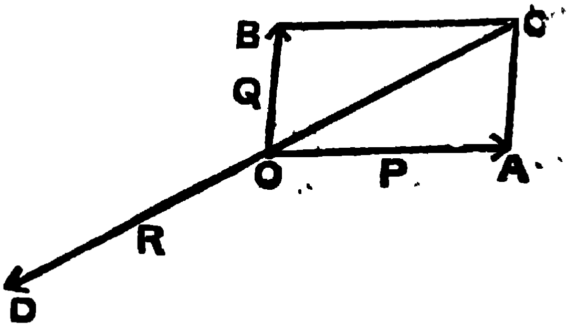
This may be proved by the method of Article 21.

164. *If three forces acting on a particle be in equilibrium, the forces are severally proportional to the sines of the angles included between the directions of the other two forces.*

Let P, Q, R, acting on O, be in equilibrium.

Let OA, OB, $OD \equiv$ these forces in magnitude and direction. Complete the parallelogram $OACB$.

By Article 130, OC represents the resultant of OA, OB. Now the particle is at rest, therefore OC must be equal and opposite to OD. (Art. 160.)



And therefore DO and OC must be in the same straight line.

By Trigonometry:

$$OA : AC : CO = \sin ACO : \sin COA : \sin OAC,$$

$$\text{Now } \sin ACO = \sin COB = \sin(180^\circ - BOD) = \sin BOD,$$

$$\sin COA = \sin(180^\circ - DOA) = \sin DOA,$$

$$\sin OAC = \sin(180^\circ - AOB) = \sin AOB;$$

$$\therefore P : Q : R = \sin BOD : \sin DOA : \sin AOB,$$

$$\text{or } \frac{P}{\sin \widehat{QR}} = \frac{Q}{\sin \widehat{RP}} = \frac{R}{\sin \widehat{PQ}}$$

where the symbol $\widehat{QR}$ denotes the angle between the directions of Q and R, etc.

The theorem established in this Article is known as *Lami's Theorem.*

THE POLYGON OF FORCES.

165. Statement.—*If any number of forces acting on a particle be represented in magnitude and direction by the sides of any closed figure taken in order, they will be in equilibrium.*

This may be proved by the method of Article 22.

The *converse* of this theorem is not necessarily true, for the reason given in Article 24.

THE PARALLELOPIPED OF FORCES.

166. Statement.—*If three forces, acting on a particle, be represented in magnitude and direction by the three adjacent edges of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal drawn through the particle.*

This may be proved by the method of Article 25.

If the forces P, Q, S act each at right angles to the plane of the other two, and if R be their resultant, then—

$$R^2 = P^2 + Q^2 + S^2.$$

RESOLUTION OF FORCES.

167. If F be any force, we may break it up into any number of pairs of components, because the straight line which represents F may be made the diagonal of any number of parallelograms. It is most convenient, however, to have the directions of the components at right angles to each other. And when this is the case, each is called *the* Component, or the 'Resolved part,' of the given force in that direction. (*See* Art. 28.)

If F denote any force acting at O in the direction OC (*see* Fig. of Art. 29), then—

Component along $OX = F \cos \theta$.

Component along $OY = F \cos (90^\circ - \theta) = F \sin \theta$.

168. *To find the resultant of any number of forces acting on a particle in the same plane.*

This may be found—

(1) By repeated applications of the Parallelogram of Forces. (*See* Art. 30.)

(2) By the Polygon of Forces. (*See* Art. 30.)

(3) By resolving the forces in two directions at right angles to each other, and compounding the results by means of the Parallelogram of Forces.

169. Thus let F_1, F_2, F_3, ... act on a particle O, and let their directions make angles α_1, α_2, α_3, ... with the line Ox. (*See* the Fig. in Art. 30, iii.)

Then $F_1 \equiv F_1 \cos \alpha_1$ along Ox, and $F_1 \sin \alpha_1$ along Oy.

And $F_2 \equiv F_2 \cos \alpha_2$ along Ox, and $F_2 \sin \alpha_2$ along Oy.

. . . .

If X be the Algebraical sum of the components along Ox,
and Y ,, ,, ,, ,, ,, Oy,

$$\text{Then } X = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots .$$

$$\text{And } Y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + \ldots .$$

Thus the system has been reduced to two forces, X and Y, acting at O at right angles to each other.

Let the resultant of X and Y be denoted by R, then

$$R^2 = X^2 + Y^2.$$

If θ be the angle which the direction of R makes with Ox, then

$$\tan \theta = \frac{Y}{X}.$$

The signs of Y and X will determine the quadrant in which R will act. (*See* Art. 30, Note 1.)

170. In order that there may be equilibrium, we must have $R=0$.

Thus, if in the equation $R^2=X^2+Y^2$ we have $R=0$,

we must have $X^2+Y^2=0$,

and $\therefore$ X must$=0$, and Y must$=0$, simultaneously.

(Compare Article 30, Note 3.)

Hence, in order that any number of forces acting on a particle may be in equilibrium, the two following conditions must be fulfilled :—

(1) *The Algebraical Sum of the component in any direction must*$=0$.

(2) *The Algebraical Sum of the components in a direction at right angles to the former must*$=0$.

See Examples XXVII. No. 39.

EXAMPLES—XXVII.

1. What is meant by the Triangle of Forces?

2. Can three forces, 7, 10, 20, acting on a particle, keep it at rest?

3. If three forces keep a particle at rest, show that each is proportional to the sine of the angle included between the other two forces.

4. If any number of forces act on a particle in one plane, explain how it may be ascertained by a geometrical construction whether the particle is at rest.

5. Two forces, each equal to P, act on a particle, and their resultant is $\frac{1}{2}P$. Find the angle between the forces.

6. State the conditions of equilibrium of three forces acting on a particle in the same plane.

7. What is meant by the Polygon of Forces?

Four forces, 3, 6, 9, 12, mutually at right angles in the same plane, act on a particle. Find by a geometrical construction the force which will keep it at rest.

8. If four equal forces act on a particle in a plane in directions which are severally inclined to each other at 45°, find by a geometrical construction the magnitude and direction of the force which will keep them at rest.

9. Why may a force be represented by a straight line? State and prove the proposition known as the Parallelogram of Forces.

10. The three forces which, acting in a plane, keep a particle at rest being known, write down the equations by which their inclinations to each other may be found.

11. If three forces, 5, 3, 10, act on a particle, find the *least* force which will produce rest.

12. If three forces acting on a particle be completely represented by the adjacent edges of a parallelopiped, find the line which completely represents their resultant.

13. Two forces, each equal to P, act on a particle at an angle of 120°; show that their resultant is also equal to P.

14. O is the middle point of a square, and forces 1, 2, 3, 4 respectively act at O in the direction of the four angles; find the magnitude of their resultant.

15. If forces of 10 and 17 act on a particle, find the greatest and least resultants respectively.

16. Two equal forces act on a particle, (1) at an inclination of 45°; (2) at an angle of 135°. Compare the resultants.

17. Show that forces 3, 5, 9, cannot keep a particle at rest.

18. Explain the terms *Resultant* and *Component*.

19. A force of 60 acts on a particle between two forces of 30 and $30\sqrt{3}$, making an angle of 60° with the former, and 30° with the latter; find the magnitude and direction of the force which will keep the particle at rest.

20. Two forces are in the ratio 3 : 2; find the angle between them, when their resultant is a mean proportional between them.

21. Given the resultant of two forces, their sum, and the angle between them, show how to determine the forces.

Resolve a force of 50 into two forces, whose sum shall be 75, and which shall include an angle of 120°.

22. Three forces, 47, 65, 70, act on a particle, the angle between each pair being 120°; find the magnitude and direction of their resultant.

23. Two forces, P and Q, have a resultant R, which makes an angle α with P; if P be increased by R, whilst Q is unchanged, show that the new resultant makes an angle $\frac{1}{2}\alpha$ with P.

24. Perpendiculars are drawn from any point within a rectangle upon its four sides. Find the magnitude and direction of the resultant of the forces which are completely represented by these lines.

25. $ABCDEF$ is a regular hexagon; find the resultant of the forces which are fully represented by AB, AD, AE.

26. Upon the sides AB, AC of a triangle ABC ($A = 90°$), the squares $ABDE$, $ACFG$ are described. Prove that the resultant of the two forces completely represented by BF, CD, is perpendicular to BC, and proportional to it in magnitude.

27. Three forces, 2, 3, 4, act at a point. The largest of the forces lies in direction between the other two, making with the larger an angle of 90°, and with the smaller an angle of 30°; find the resultant.

28. Forces $P-Q$, P, $P+Q$, act at a point in directions parallel to the sides of an equilateral triangle taken in order; find their resultant.

29. ABC is a triangle, in which A is 90°, and B is 60°. AD is drawn perpendicular to BC. Find the magnitude and direction of the resultant of forces fully represented by AB, AC, AD, if AD represent a weight of 3 lbs.

30. ABC, DBC are two triangles of the same altitude on the same base BC. Forces are fully represented by BA, AC, CD, BD, BC; find their resultant.

31. If eight forces acting on a particle be fully represented by lines drawn from the angles of any quadrilateral to the middle points of the opposite sides, show that they will form a system in equilibrium.

32. $ABCD$ is a square. A force of 5 acts from A to B, 4 from B to C, 7 from C to D. Find the single force which will maintain equilibrium.

33. Four forces are completely represented by AB, AD, CB, CD, and act on a quadrilateral $ABCD$. Show that their resultant is completely represented by four times the line joining the middle points of the diagonals.

34. ABC is a triangle, D a point in BC such that $BD : DC = 1 : 3$. Prove that the resultant of AC and 3 times AB is 4 times AD.

35. Three forces acting on a particle are represented in magnitude and direction by the sides of a triangle *ABC*. Find the magnitude and direction of the resultant (1) when the directions are *AB*, *BC*, *CA*; (2) when the directions are *BA*, *BC*, *AC*.

36. The circumference of a circle is divided into a number of equal parts, and from one of the points of division straight lines are drawn to the rest. Show that the line of action of the resultant of the forces which these lines represent will coincide with the diameter through that point.

37. If there be eight points taken as in Ex. 36, show that the resultant is represented by 4 times the diameter.

38. If *ABCD* be a parallelogram, and four forces acting at *A* towards the middle points of the four sides be represented by the lines joining *A* with those middle points, find the magnitude and direction of their resultant.

39. In Art. 170, show that, for equilibrium, the algebraical sums of the resolved parts of the forces in *any* two directions must separately vanish.

40. *ABCD* is a quadrilateral, the angles *A* and *C* being right angles. Forces *AB*, *AD* act at *A*, forces *CB*, *CD* act at *C*. Show that their resultant bisects the angle *AEC*, *E* being the middle point of *BD*.

41. Three forces act in a plane at a point. If one of the forces be given in magnitude and direction, and a second force in direction only, prove that the least force which will produce equilibrium is at right angles to the second force.

42. Forces 18 and 30 act at a point along lines at right angles to one another, and a force 6 acts along a line between them and inclined at 30° to the first force. Prove that their resultant is 40·2, and makes with the 18 an angle $= \tan^{-1}(1{\cdot}4)$.

43. Forces 4, $10\sqrt{2}$, 14 act at a point E., N.E., N. respectively. Prove that their resultant is 27·8, and makes with the force 4 the angle $\tan^{-1}(1\frac{4}{7})$.

44. Forces 20, 25, 30 act at a point. The angle between the first and second is 45°, and that between second and third is 75°. Prove that their resultant is 49·15, and that its direction with the first force is $\tan^{-1}(1{\cdot}9)$.

CHAPTER VII.

TWO PARALLEL FORCES.

171. BEFORE proceeding further, it is necessary to explain certain terms, and to make certain assumptions.

172. If any force be applied to a rigid body we assume as an axiom that its effect is not changed if made to act at any other point of the body *in the line of action* of the force.

This is known as the principle of the **Transmissibility of Force.** It can be verified sufficiently by many direct experiments.

173. We assume as an axiom that if two *equal* and *opposite* forces be *introduced into* a system of forces acting on a body, or *removed from* such a system, the system of forces will not be disturbed. (*See* Art. 161.)

174. If a force act on a body *by means of a string*, we assume that if the string be placed over a smooth peg or pulley, the effect of the force is transmitted without change of *magnitude*. The object sought in such a case is merely to change the *direction* of the force.

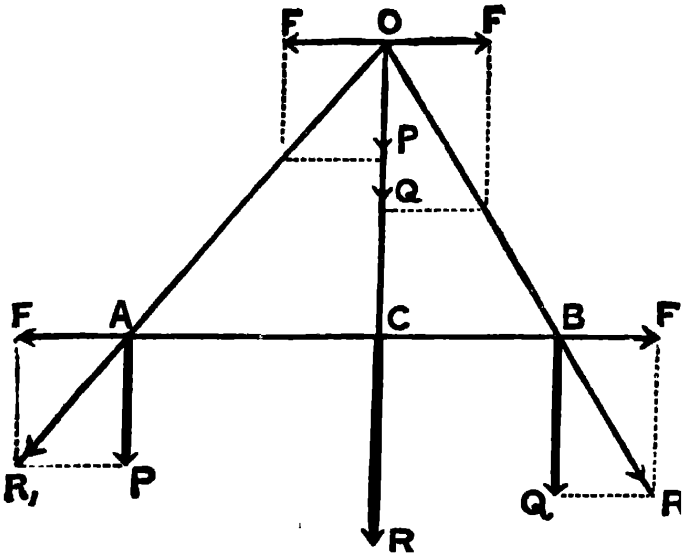
175. A force acting along a rod tending to *compress* it is called a *Thrust;* if tending to stretch it the force is called a *Tension.*

A force acting along a string is always known as a *Tension.*

176. Parallel forces are said to be '**like**' if they have the same direction; and to be '**unlike**' if they have opposite directions.

177. *To find the Magnitude and Direction of the Resultant of two like parallel forces acting on a body.*

Let A and B be the points where the forces P and Q act. At A and B apply two equal and opposite forces F acting in the line AB. These will balance, and will not disturb the system. (Art. 173.)



Complete the parallelograms FP and FQ.

Then P and F at $A \equiv R_1$ at A. (Art. 130.)

And Q and F at $B \equiv R_2$ at B.

Let the lines of action of R_1 and R_2 meet in O. (Art. 172.)

At O resolve R_1 into components F and P parallel to their former directions;

and at O resolve R_2 into components F and Q parallel to their former directions.

The equal and opposite forces F at O balance, and may be removed. The Resultant is $P+Q$, and OC, the line of action of P and Q acting at O, is parallel to the lines of action of P and Q acting at A and B respectively.

If R be the resultant of P and Q,

$$R=P+Q; \quad \ldots\ldots\ldots \quad (1.)$$

and R may be supposed to act at C. (Art. 172.)

To determine the position of C.

The triangles OCA, APR_1 being equiangular, are similar;

$$\therefore \frac{P}{F}=\frac{OC}{CA}. \qquad \text{(Euc. vi. 4.)}$$

And the triangles OCB, BQR_2 being equiangular are similar;

$$\therefore \frac{Q}{F}=\frac{OC}{CB}.$$

Dividing the first of these results by the second, we have

$$\frac{P}{Q}=\frac{CB}{CA}. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

We therefore conclude that when two like parallel forces act on a body:—

1. *The Resultant acts in the same direction as the forces.*
2. *It is equal to the sum of the forces.*
3. *It acts at a point which divides the line joining their points of application inversely as the forces.*

The Student will notice that the position of C depends on the magnitudes only of P and Q.

178. *To find the Magnitude and Direction of the Resultant of two unlike parallel forces acting on a body.*

Let A and B be the points at which P and Q act; Q being the greater force.

At A and B apply two equal and opposite forces F, acting in the line AB. These will balance, and their introduction does not disturb the system.

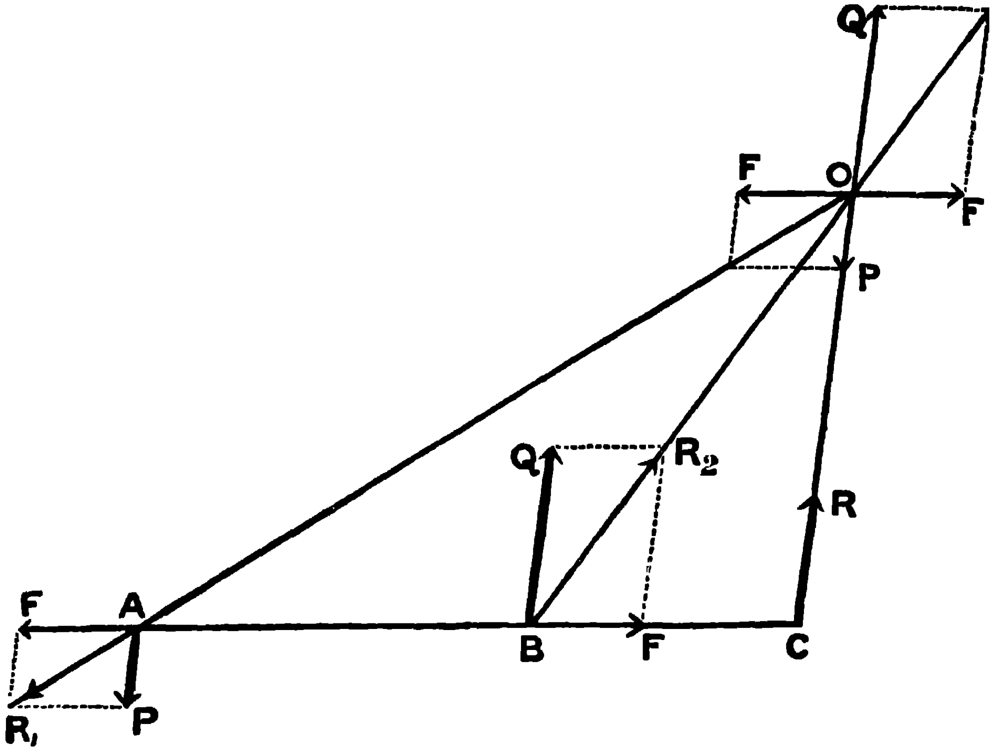
Complete the parallelograms FP and FQ.

$$\text{Then } P \text{ and } F \text{ at } A \equiv R_1 \text{ at } A;$$
$$\text{and } Q \text{ and } F \text{ at } B \equiv R_2 \text{ at } B.$$

Let the lines of action R_1 and R_2 meet in O.

At O resolve R_1 into components F and P parallel to their former directions.

And at O resolve R_2 into components F and Q parallel to their former directions.



The equal and opposite forces F at O balance, and may be removed.

The Resultant of P and Q acting in *opposite* directions is $Q-P$, and OC, the line of action of P and Q acting at O, is parallel to the lines of action of P acting at A and Q acting at B. Hence if R be the resultant of P and Q, we have

$$R=Q-P, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

and R may be supposed to act at C in AB produced.

To determine the position of C.

The triangles OCA, APR_1 being equiangular, are similar;

$$\therefore \frac{P}{F}=\frac{OC}{CA}.$$

The triangles OCB, R_2FB being equiangular, are similar;

$$\therefore \frac{Q}{F}=\frac{OC}{CB}.$$

Dividing the first of these results by the second, we have

$$\frac{P}{Q}=\frac{CB}{CA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.)$$

We therefore conclude that when two unlike parallel forces act on a body:—

1. *The Resultant acts in the direction of the greater force.*
2. *It is equal to the difference of the forces.*
3. *It acts at a point which divides* externally *the line joining their points of application produced inversely as the forces.*

Note.—The point (C) at which the resultant acts is always in AB produced through the point of application of the *greater force*.

179. From the result (2) of the preceding Article we have—

$$P \,.\, CA=Q \,.\, CB\,;$$
$$\therefore\ P(AB+BC)=Q \,.\, BC\,;$$
$$\therefore\ BC(Q-P)=P \,.\, AB\,;$$
$$\therefore\ BC=\frac{P \,.\, AB}{Q-P}$$

Now when Q is nearly equal to P, BC becomes very great when compared with AB.

As a *Special Case* let $Q=P$.

$$\text{Then } BC=\frac{P \,.\, AB}{0}$$
$$\therefore\ BC=\infty$$

and also from (1) $R=0$.

The student will see that in this case R_1A and BR_2 (Fig. Art. 178) are parallel lines, and therefore the point O is not situated at a finite distance.

If two *equal unlike parallel* forces act on a body, the system is called a **Couple**, and the two results $R=0$, and $BC=\infty$, indicate that when a Couple acts on a body it cannot be replaced by any single force acting at any finite distance.

NOTE.—The relation established in Articles 177 and 178 between the Forces and their distances from the point at which their Resultant acts is most conveniently written,

$$P.AC=Q.BC.$$

180. The following simple method of obtaining the results of Art. 178 from those of Art. 177 deserves the student's notice. If in the fig. of Art. 177 we introduce at C a force R_1 equal and opposite to $R(=P+Q)$, we shall have three parallel forces in equilibrium, viz. P, R_1, Q. And it is evident that each of these forces is *equal* and *opposite* to the *resultant* of the other two forces.

Now if we regard P and R_1 as the given *unlike* parallel forces we may conclude at once:—

(1) Their resultant is equal to Q, and therefore is equal to R_1-P.

(2) Their resultant acts at B opposite to Q, and therefore acts in the direction of R_1, the greater force.

(3) By Art. 177, that $\dfrac{P}{R_1-P}=\dfrac{CB}{CA}$;

$$\therefore P.CA=R_1.CB-P.CB;$$
$$\therefore P(CA+CB)=R_1.CB;$$
$$\therefore \frac{P}{R_1}=\frac{CB}{AB};$$

and these are the results of Art. 178.

Thus the fig. and the method of Art. 178 can be dispensed with.

Example i.—Two like parallel forces, 10 and 15, act at points 21 inches apart; find the magnitude and point of application of their resultant. By equation (1) Art. 177, $R=P+Q$,

$$\therefore R=25.$$

Let x = distance from 10 at which R acts ($= AC$ in Fig. Art. 177),

$$\therefore\ 10x = 15(21 - x).$$

From which equation, $x = 12{\cdot}6$ inches.

Example ii.—Two unlike parallel forces, 3 and 12, act at points 12 inches apart; find the magnitude and point of application of their resultant. By equation (1) Art. 178, $R = Q - P$;

$\therefore\ R = 12 - 3 = 9$ in the direction of the 12.

Let BC (Fig. Art. 178) $= x$.

Then if $P = 3$, and $Q = 12$, we have

$$3(12 + x) = 12x,$$

From which equation, $x = 4$ inches.

Example iii.—Break up a force of 8 into two like parallel forces acting at points 4 feet apart, so that one of them may act at a distance of 1 foot from the given force.

Let the latter force $= P$; then the other $= 8 - P$.

In the Fig. of Article 177, let $AC = 1$ ft.; then $BC = 3$ ft.

Therefore $P \times 1 = (8 - P) \times 3$.

From which equation, $P = 6$;

$\therefore$ the other force $= 2$.

Example iv.—The resultant of two unlike parallel forces which act at points 12 inches apart is 10, and it acts at a point 4 inches from the greater force; find the magnitude of the forces.

We may use the Fig. of Article 178.

Let P be the smaller force; then $P + 10$ = the greater;

Therefore $P(12 + 4) = (P + 10) \times 4$.

From which equation, $P = 3\frac{1}{3}$;

$\therefore$ the other Force $= 13\frac{1}{3}$.

EXAMPLES—XXVIII.

1. Find the magnitude and point of application of the resultant of two like parallel forces of 7 and 5 acting at points 24 inches apart.

2. Find the magnitude and point of application of the resultant of two unlike parallel forces of 10 and 6 acting at points 8 inches apart.

3. Two unlike parallel forces, 52 and 78, act at points 16 feet apart; find the magnitude and position of their resultant.

4. Find the resultant of two like forces whose lines of action are parallel.

5. Two parallel forces, 3 and 7, act in opposite directions; the distance between their points of application is 16 inches; find the magnitude and point of application of their resultant.

6. Explain the result when two equal and unlike parallel forces act on a rigid body.

7. Deduce from the results of Example 4 the magnitude and direction of the resultant in the case of unlike parallel forces.

8. Two like parallel forces, 16 and 24, act at distances of 12 and 25 respectively to the right of a fixed point; at what distance does their resultant act?

9. Two unlike parallel forces, 82 and 15, act at distances 12 and 30 to the right of a certain point in their plane; find the distance of the resultant from the same point.

10. If the resultant of two unlike parallel forces be 12, and act at distances of 6 in. and 9 in. respectively from the lines of action of the forces, find the forces.

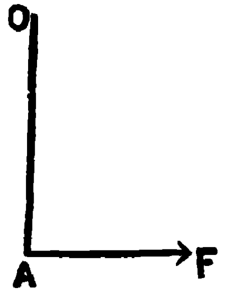
11. Break up a force of 40 into two like parallel forces 3 feet apart, one of them being 16 inches from the given force.

12. Break up a force P into two like parallel forces in the ratio $m:n$, if one act at a distance a from P, find the distance at which the other force acts from P.

CHAPTER VIII.

MOMENTS.

181. If a rod *OA* be capable of rotation about a fixed point *O* in it, and a force *F* act on the rod at the point *A*, then it is evident that the rod will turn round *O* under the action of *F*, unless the direction of the force pass through the fixed point *O*. In this case the reaction of the point will keep the force in equilibrium.



182. The measure of the power of a force to produce rotation will evidently depend on (1) the magnitude of *F*, and (2) the distance of its line of action from the point *O*. We may therefore say that the *Measure* of the Power of a force to produce rotation about any point is the *product of the Force by its perpendicular distance from the point, certain units of force and length having been selected.*

This product is called the **Moment of the Force** with respect to the point.

Hence, if the Measure of a Force be *F*, and *p* be the distance of the point from its line of action, the *Measure of the Moment* will be $F.p$.

183. The physical idea involved in a moment is a *Twist*. If we consider rotation in one direction *positive*, then rotation in the opposite direction will be *negative*. It is usual to regard rotation *in the direction of the hands of a watch* as negative.

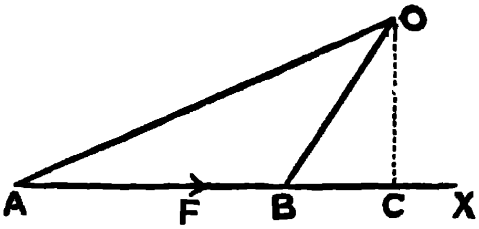
In the Figure (Art. 181) the moment is considered to be positive. If a body be at rest under the turning tendency of two forces, these tendencies must be equal, and therefore the algebraical sum of the moments will be zero.

184. *To represent the Moment of a Force with respect to a point Geometrically.*

If O be a fixed point in a rigid body, and a force F act in the direction of the line AX.

Make $AB \equiv F$; draw OC perpendicular to AX.

Then, Moment of F about $O = F.OC = AB.OC = 2\triangle AOB$.



Hence the *Numerical* Measure of the Moment of a force about a point will be represented by *twice the area of the triangle whose base is the line which represents the force in magnitude and direction and whose vertex is at the point.*

185. If O be in the line of action of the force F, the triangle vanishes, and then the Moment is zero. Therefore we infer that the *Moment of a force about any point in its own line of action vanishes*; and, conversely, if the *Moment of a force about any point vanishes, the point is in the line of action of the force.*

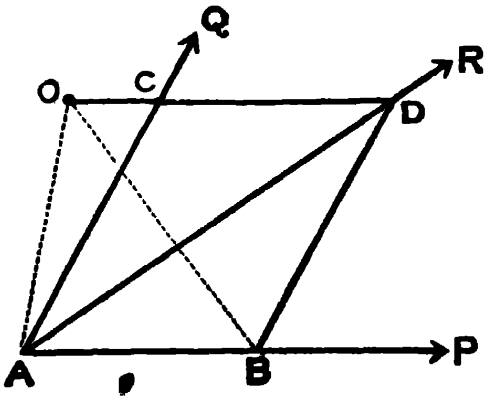
186. THEOREM.—*If two forces act on a body, the Algebraical Sum of their Moments about any point in their plane is equal to the Moment of their Resultant about that point.*

Case i.—*When the Forces act at a point.*

Let AP and AQ be the *directions* of two forces acting at A, and AR the *direction* of their Resultant.

Let O be any point in their plane. Through O draw OD parallel to AP, meeting AQ and AR in C and D respectively.

Through D draw DB parallel to AQ;



Then AB, AC, AD will *completely* represent P, Q, R respectively.

Join OA and OB.

Then the Moment of P about $O = 2\triangle AOB$ (Art. 184)

$= \square ABDC$ (Euc. i. 41) $= 2\triangle ACD = 2\triangle AOD - 2\triangle AOC$

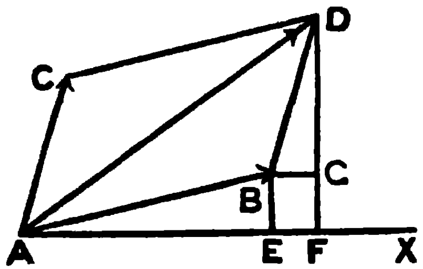
$=$ Moment of R about $O -$ Moment of Q about O;

$\therefore$ Moment of P about $O +$ Moment of Q about O

$=$ Moment of R about O. (Q.E.D.)

The student ought to prove this Theorem when the point is between AB and AC.

187. The Proposition may also be established in the following way. We must first show that the resolved part in any direction of the Resultant of two forces acting at a point is equal to the sum of the resolved parts of the Components in the same direction.



To show that the component of AD along $AX =$ comp. of AB along $AX +$ component of AC along AX.

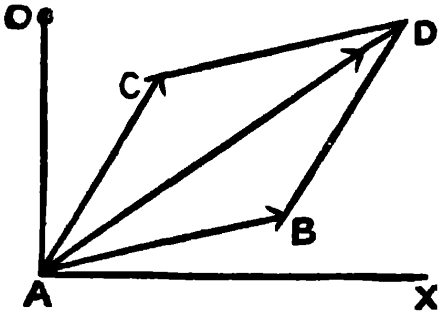
Draw BE and DF perpendicular to AX, and draw BG parallel to AX.

Component of AD along $AX = AF$; (Art. 29)

" AB " $AX = AE$;

" AC or BD " $AX = EF (= BG)$

Now $AF = AE + EF$. (Q.E.D.)



To prove the Proposition: Let AD be the Resultant of the forces AB and AC. Join AO, and draw AX at right angles to AO.

$$AD \cos XAD = AB \cos XAB + AC \cdot \cos XAC.$$

Multiply each term by AO, and change to the complements of the angles;

$\therefore AD \cdot AO \sin DAO = AB \cdot AO \sin BAO + AC \cdot AO \sin CAO$;

$\therefore$ *by Trigonometry*, $2\triangle AOD = 2\triangle AOB + 2\triangle AOC$,

or, Mom. of AD about $O =$ Mom. of AB about $O +$ Mom. of AC about O.

(Q.E.D.)

188. Case ii.—*When the Forces are parallel.*

Draw OB perpendicular to the lines of action of the forces P, Q and their resultant R.

Then $P.AC = Q.BC$ (Art. 180 Note).

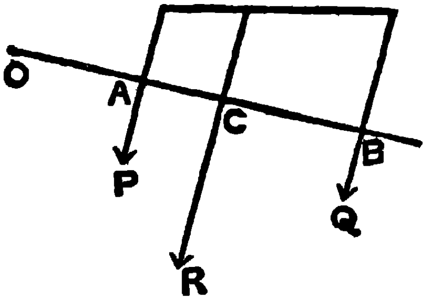
$\therefore P(OC - OA) = Q\,(OB - OC).$

$\therefore P.OC - P.OA = Q.OB - Q.OC.$

$\therefore (P + Q).OC = P.OA + Q.OB,$

but $P + Q = R$ (Art. 178).

$\therefore R.OC = P.OA + Q.OB.$ (Q.E.D.)



As an exercise the student may establish the Proposition (1) when O is between P and Q; (2) when P and Q are unlike.

189. This Proposition, being true for *all* positions of the point, must be true when it is in the line of action of the resultant. The Moment of the resultant in this case vanishes. (Art. 185.)

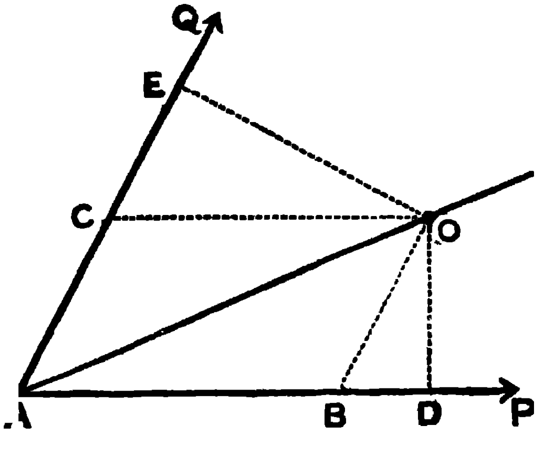
Therefore '*The Moments of two Forces about any point in the line of action of their Resultant are equal and opposite.*'

190. The converse is also true, viz.:—'*If the Moments of two Forces about any point in their plane be equal and opposite, that point must be in the line of action of their Resultant.*'

Let AP and AQ represent the directions of the forces P and Q, and let O be a point in the plane such that $P.OD = Q.OE$.

Then shall O be in the line of action of the resultant.

Through O draw OB parallel to AQ, and OC parallel to AP. Then, by Data, $P.OD = Q.OE$.



$$\therefore \frac{P}{Q} = \frac{OE}{OD} = \frac{OC}{OB} = \frac{AB}{AC}.$$

Therefore AB and AC represent the magnitude of P and Q respectively, and AO represents their resultant in magnitude and direction.

Therefore O is in the line of action of the resultant of P and Q. (Q.E.D.)

EXAMPLES—XXIX.

1. What is meant by the moment of a force with respect to a point?

2. The line of action of a force of 10 lbs. weight passes at a distance of 8 inches from a fixed point. What is the measure of the moment about that point, a pound weight being the unit of force, and an inch the unit of distance?

3. A force of 12 acts along a median of an equilateral triangle whose side is 18; find the measure of its moment about each angle of the triangle.

4. A force of 6 acts along one side of an equilateral triangle whose side is 10; find the measure of its moment about the opposite angle.

5. A force of 20 acts along a diagonal of a square whose side is $8\sqrt{2}$; find the measure of its moment about each of the four angles.

6. The lines of action of two like parallel forces of 10 and 8 respectively are distant 4 and 12 from a fixed point; compare their moments about the point.

7. Where must the point in Ex. 6 be situated in order that the moments may be equal and opposite?

8. If two forces meet in a point, prove that their moments are equal about any point in the line of action of the resultant.

9. Show that the moment of a force about a point is represented geometrically by twice the area of the triangle whose base is the line representing the force in magnitude and direction, and whose vertex is at the point.

10. The algebraical sum of the moments of two forces meeting in a point about any point in their plane is equal to the moment of their resultant about the same point.

11. Prove the last example when the point is *within* the lines of action of the forces.

12. If three forces acting in the same plane at a point be in equilibrium, the moment of any one of them about a point in their plane is equal and opposite to the sum of the moments of the other two about that point.

13. A man can exert a force equal to the weight of 224 lbs., and pulls on a rope fastened to the top of a post, the rope being twice the length of the post. What horizontal force applied at the middle of the post will keep it from falling?

14. At what point of a tree must one end of a rope of given length a be fixed, so that a man pulling at the other end may exert the greatest force to pull it over?

15. The connecting-rod of an engine is inclined to the crank-arm at an angle of 30°. Compare the moment of the force to turn the shaft when in this position with the greatest moment when in the most favourable position.

191. It is very important to extend the Theorem of Article 186 to any number of forces acting in a plane.

Statement of **Varignon's Theorem of Moments.**

If any number of forces act in a plane on a body, the Algebraical Sum of their Moments about any point in the plane is equal to the Moment of their Resultant about that point.

In Articles 186 and 188, the Theorem was proved for two forces. If we consider their resultant and a third force in the system as constituting another pair, the Theorem will hold for them, and therefore it holds for three of the forces, and so on for any number of forces. (Q.E.D.)

192. Symbolical Statement of Varignon's Theorem of Moments.

Let the distances of the selected point from the lines of action of the forces P, Q, S, T . . . . . and their resultant R be p, q, s, t, ... and r respectively; then we shall have—

$$Rr = Pp + Qq + Ss + Tt + , \ldots\ldots$$

If any of the forces act in the negative direction, we must write its moment with a *negative* sign, and R will then be the *Algebraical Sum* of the forces.

193. The student will notice that, *if the point be situated in the line of action of the resultant, then the Algebraical Sum of the Moments will vanish;* and, conversely, *if the Algebraical Sum of the Moments of any number of forces acting on a body vanishes about a point, the line of action of the resultant passes through that point.* (*See* Articles 189 and 190.)

Example i.—Five vertical forces, 9, 12, 18, 6, 5, act at points 3 feet apart on a straight horizontal rod; where does their resultant act?

Let x = distance from the line of action of the first force.

$R = 50$.

Taking Moments about the point at which 9 acts (Art. 192)—

Then $50x = 9 \times 0 + 12 \times 3 + 18 \times 6 + 6 \times 9 + 5 \times 12$;

$\therefore 50x = 258$; $\therefore x = 5 \cdot 16$ feet.

Example ii.—Two parallel forces, 8 and 5, act in opposite directions at distances 3 and 11 to the right of a fixed point. Find at what distance their resultant acts.

Let x = distance from the fixed point *measured to the right.*

$R = 3$ in the direction of the 8 (Art. 178).

Taking Moments about the fixed point—

Then $3 \times x = 8 \times 3 - 5 \times 11$.

From which $x = -10\frac{1}{3}$ to the right,

or $= 10\frac{1}{3}$ to the left of the fixed point.

Example iii.—Parallel forces 3, 5, 6, 4 act at equal distances of 8 inches along a rod; where must a prop be placed in order that the rod may rest in a horizontal position?

It is evident that the prop must be placed at the point where the resultant acts.

Let x = distance from 3.

$R = 18$.

Taking Moments about the point at which 3 acts (Art. 192)—

Then $18x = 3 \times 0 + 5 \times 8 + 6 \times 16 + 4 \times 24 = 232$;

$\therefore x = 12\frac{8}{9}$ inches from the force 3.

EXAMPLES—XXX.

1. Four parallel forces, 10, 14, 6, 25, act at distances 2, 4, 9, 3 from the end of a horizontal rod; where does their resultant act?

2. Four parallel forces, 3, 6, 9, 12, are placed at equal distances along a straight horizontal rod; show that the resultant will act at the same point when the 9 is removed.

3. Five parallel forces, 1, 6, 3, 4, 8, act 1 foot apart on a straight horizontal rod; what force must be added to the 1, in order that if the rod be supported where the 3 acts, it may remain horizontal?

4. A rod is 6 feet long; parallel forces, 6, 12, 18, act at the ends and middle respectively, and at right angles to the rod; find the magnitude and point of application of the force which will produce equilibrium.

5. Prove that the algebraical sum of the moments of any number of parallel forces, acting in the same plane on a body about a point in the plane, is equal to the moment of their resultant about that point.

6. Three parallel forces, 10, -15, 40, act at points 3 feet, 4 feet, 5 feet respectively from one end of a rod and at right angles to the rod; where does their resultant act?

7. Four parallel forces, 1, 6, -9, 8, act at points 3 inches apart along a straight rod and at right angles to the rod; where must the rod be supported in order that it remain in equilibrium?

8. A uniform straight rod, made up of two lengths a and b, has vertical forces Wa and Wb acting from the middle points of a and b respectively; where must the rod be supported to remain in a horizontal position?

9. A uniform rod has a vertical force W acting at its middle point, and when suspended at a certain point rests in a horizontal position with vertical forces W_0 and W_1 at its extremities, or W_2 and W_0 at the same ends. What vertical force at one end will keep it horizontal?

10. A uniform rod $ABCD$, 2 feet long, has a vertical force of 15 acting at its middle point. It rests on two props, B and C, 10 inches apart, and the pressure on C is 4 times that on B; find the greatest vertical force that can act at D without overturning the rod.

11. A uniform rod 20 inches long rests in a horizontal position on a smooth peg, and has vertical forces of 5 and 3 at distances of 4 inches and 5 inches from one end; how far is the peg from that end, if an additional vertical force of 2 act at the middle of the rod?

12. Four parallel forces, 3, 2, 5, 7, act at distances 6 inches apart along a straight rod and at right angles to the rod; where must a force of 17 act in order to maintain equilibrium?

13. Vertical forces of 6 and 3 at the ends of a horizontal rod (length l), and W at a certain point in it, have their resultant acting at the middle of the rod; find the point at which W acts.

14. If, in Ex. 13, W act at a distance of one-fourth of the rod from the smaller force, find W.

15. A uniform rod, 13 feet long, has a vertical force of $2\frac{1}{2}$ acting downwards at its middle point. The rod can turn about one end. If the other end be kept up by a vertical force of 9, where must a vertical force of 60 act if the rod is to remain horizontal?

CHAPTER IX.

CENTRE OF GRAVITY.

194. In Article 177 it was proved that if two parallel forces P and Q be supposed to act at the points A and B, the line of action of their resultant will divide AB at C inversely as the forces.

The *position* of C will evidently not be altered if the lines of action of the forces be changed through any angle at their points of application, provided only that they are still *parallel.* (*See* Art. 187 at end.)

This point is known as the *Centre* of the two parallel forces.

The position of the point at which the resultant of any number of parallel forces having fixed points of application acts, is not affected by all the forces, while still remaining parallel, being deflected at their points of application through any angle. We therefore arrive at the following definition:—

Def.—*The Centre of any number of parallel forces having fixed points of application is the point through which the direction of their resultant passes, whatever be the directions of the parallel forces.*

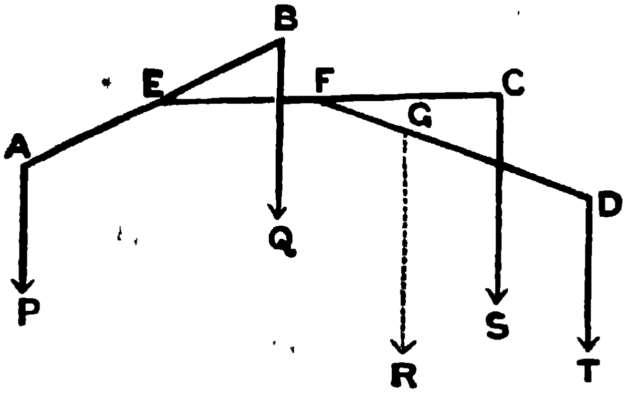
This is very important.

195. *To find the Centre of any number of Parallel Forces.*

Let parallel forces, P, Q, S, T act at A, B, C, D respectively.

Join AB, and divide it at E, so that $AE : EB = Q : P$.

Then $P+Q$ acts at E (Art. 177), and S acts at C.
Divide EC at F, so that $EF:FC=S:P+Q$.



Then $P+Q+S$ acts at F, and T acts at D.
Divide FD at G, so that
$FG:GD=T:P+Q+S$.
Then $P+Q+S+T$ acts at G.

Now the position of G is not affected by the direction of the Parallel Forces.

∴ by Def. G is *centre of the Parallel Forces.*

In this way we have shown that these four Parallel Forces have a centre, and we have also found its position; and similarly for any number.

Note.—In this may also be found the centre of gravity of a number of heavy particles. (Art. 203.)

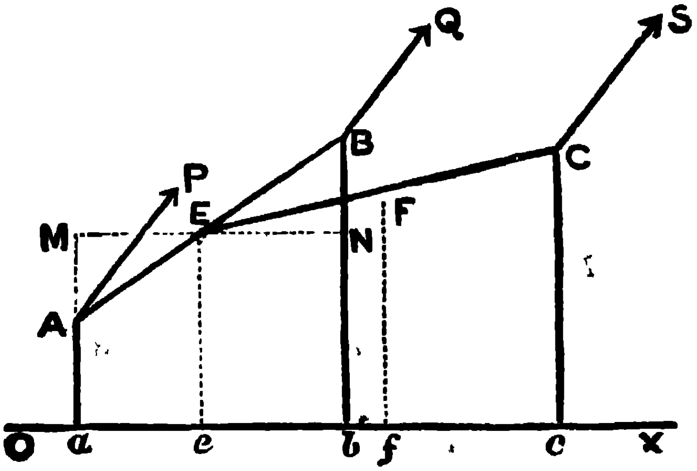
196. *To find the distance of the Centre of a system of Parallel Forces from any line in the plane containing the points of application of the forces, having given the magnitude of the forces and the distances of their points of application from the line.*

Let OX be any straight line in the plane of the forces.

Let P, Q, S, ... act at A, B, C, ... at distances p, q, s, ... from OX.

Divide AB at E so that $AE:EB=Q:P$.

Through E draw MN parallel to OX, meeting Aa, Bb in M and N.



$$\text{Then } \frac{P}{Q}=\frac{EB}{EA}=\frac{BN}{AM}=\frac{Bb-Nb}{Ma-Aa}=\frac{q-Ee}{Ee-p};$$

$$\therefore P.Ee-Pp=Qq-Q.Ee;$$

$$\therefore (P+Q)Ee=Pp+Qq.$$

In the same manner, when we have $P+Q+S$ acting at F (Art. 195), then

$$(P+Q+S)Ff=(P+Q)Ee+Ss,$$

$$\text{or, } (P+Q+S)Ff=Pp+Qq+Ss.$$

And similarly for any number of Parallel Forces.

If y be the distance from OX of the Centre of any number of Parallel Forces,

$$\text{Then } y=\frac{Pp+Qq+Ss+Tt+\ldots\ldots}{P+Q+S+T+\ldots\ldots}$$

Similarly the distance (x) of the 'Centre' from a line Oy at right angles to OX can be found, and thus the position of the 'Centre' in the plane can be determined.

NOTE.—If some of these forces act in the opposite direction, we must remember that we are dealing with *Algebraical Sums* in both the numerator and denominator of this result.

Example i.—If four like parallel forces, 2, 3, 5, 7, act at points distant 3, 4, 6, 12 inches from a straight line in the plane, find the distance of their centre from that line.

We use the result of Article 196. Let r be the distance required.

$$\therefore\ (2+3+5+7)\,r=2\times3+3\times4+5\times6+7\times12;$$

$$\therefore\ 17r=6+12+30+84.$$

From which $r=7\tfrac{13}{17}$ inches.

Example ii.—If five parallel forces $7, -3, -10, 2, 8$ act at points distant 5, 10, 3, 4, 6 inches from a straight line in the plane, find the distance of their centre from that line.

See Note to Article 196.

$$(7-3-10+2+8)\,r=7\times5-3\times10-10\times3+2\times4+8\times6;$$

$$\therefore\ 4r=35-30-30+8+48.$$

$$\therefore\ 4r=31.$$

Therefore $r=7\tfrac{3}{4}$ inches.

197. The weight of a body is the force of the earth's pull on its mass, and is therefore the resultant of the weights of its various particles.

These weights form a system of like parallel forces, and by the method of Article 195 we can determine their Centre. The centre in this case is called the **Centre of Gravity.**[1]

DEF.—*The Centre of Gravity of a body is that point through which the resultant of the weights of all its particles acts in whatever position the body is placed.*

From this it follows—(1) that the whole weight of a body may be supposed to act at its C. G.; and (2) that if the C. G. be supported the body will rest in any position.

198. *If a body be suspended at any point, its C. G. will be in the vertical line through the point of suspension when the body is at rest.*

Let G be the C. G. of a body capable of turning round a fixed point O.

Let $GC \equiv$ its weight acting vertically through G.

Resolve GC into GB in the direction OG, and GA at right angles to it.

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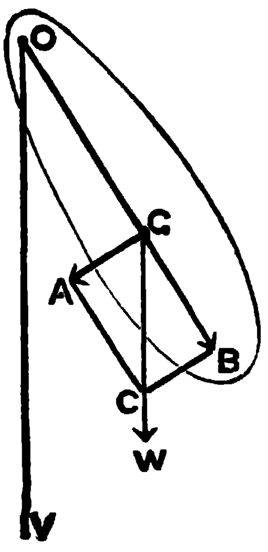
[1] Strictly this point is the Centre of Mass.

The component *GB* is kept in equilibrium by the reaction of the fixed point *O*.

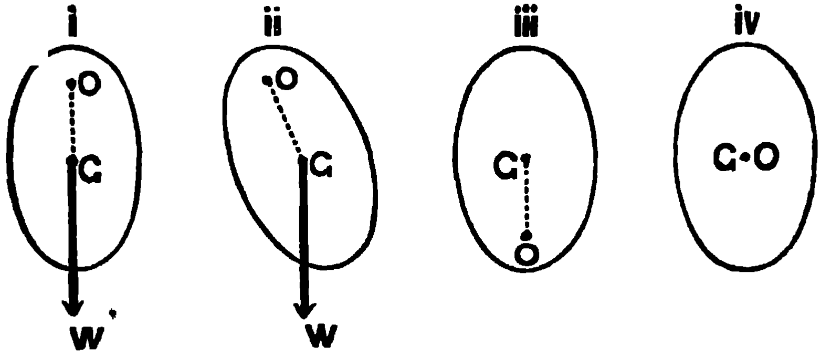


There remains the component *GA*, which will cause the body to move towards the vertical line *OV*; and it is evident that such a movement will always take place when *G* is not in the vertical line *OV*.

199. From this we can determine experimentally the position of the C. G. of a plane surface of irregular shape. Suspend the lamina successively from two points of its edge; if, in each case, a vertical line be drawn on its surface passing through the point of suspension, the intersection of these lines will be the C. G. required.

The points of suspension must of course be so chosen that the lines drawn on the surface do not coincide.

STABLE, UNSTABLE, AND NEUTRAL EQUILIBRIUM.



200. Let a body be capable of rotating about a fixed point *O*, and suppose *G* to be its C. G.

In Fig. i., because *G* is in the vertical line through *O* and *below* *O*, the body is at rest.

In Fig. ii., where the body has been slightly displaced, it will tend to return to the position in Fig. i. (*See* Art. 198.)

In Fig. iii., where G is in the vertical line through O and *above* O, the body is at rest; but if the body be slightly displaced it will tend to turn round O, and will take up the position in Fig. i.

In Fig. iv., where O and G coincide, the C. G. is supported, and therefore the body will be at rest in any position whatever. (*See* Art. 197.)

201. Keeping these considerations in view when a body in a position of equilibrium is slightly displaced—(1) If the effect of the forces in this new position be to restore the body to its old position, the equilibrium is said to be **Stable**; (2) If the effect be to make it move farther from its old position, the equilibrium is said to be **Unstable**; (3) If there be no effect one way or the other, the equilibrium is said to be **Neutral.**

Examples *of Stable Equilibrium.*—A swing, a cradle, a rocking-horse, a pendulum, a ship well ballasted, a cone standing on its base on a horizontal plane.

Examples *of Unstable Equilibrium.*—A narrow boat when the occupants stand up, a high chair in which a child is seated, a cart heavily loaded atop, deck-loaded ships, a cone placed on its vertex, a lead pencil stood on end, and in fact all bodies which we call 'top-heavy.'

Examples *of Neutral Equilibrium.*—A cricket ball placed on a table, a sphere floating in a fluid, a cone or cylinder resting on its side on a horizontal plane.

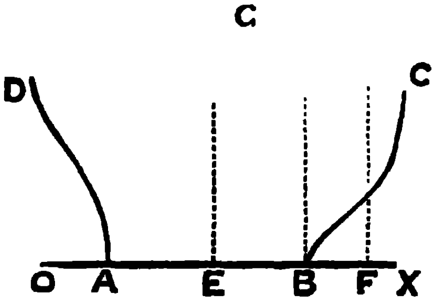
202. *When a body is placed on a horizontal plane it will stand or fall according as the vertical line through its C. G. falls within or without the base.*

Let a body $ABC \ldots D$ be placed on a horizontal plane OX.

Since its weight acts vertically through its C. G., then—

(1) If GE be the vertical through G, the weight may be

resolved into two components, one along *GA* and the other at right angles to it. The tendency of the latter component to produce rotation about *A* is *met by the reaction* of the rigid plane, and therefore the body will have no rotation about *A*. Similarly it may be shown that it will have no rotation about *B*, and therefore the body will stand.



(2.) If *GF* be the vertical through *G*, the component of the weight at right angles to *GA* will as in (1) produce no rotation about *A*, but its component at right angles to *GB*, *meeting no reaction*, will cause the body to topple over round the point *B*.

(3.) If *GB* be the vertical through *G*, the body will just stand, but it will be on the point of toppling over.

If the body be placed on an inclined plane, it follows that if the *vertical* through the C. G. fall—(1) *within* the base, the body will slide before toppling over; (2) *without* the base, the body will topple over before sliding; (3) *on the edge* of the base, the body will slide and topple at the same time.

203. We now proceed to find the C. G. of various bodies.

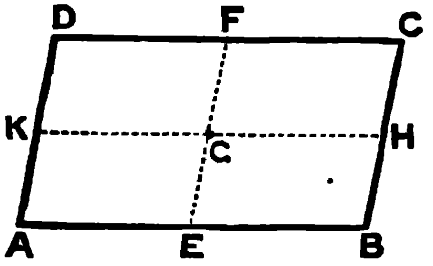
The process of finding the C. G. of a number of heavy particles is the same as finding the Centre of a system of like parallel forces. (*See* Art. 195.) We shall deal only with homogeneous bodies.

A homogeneous body is one in which matter is distributed uniformly, and therefore one in which equal volumes have equal weights. And in the case of plane figures, the weights are proportional to the areas.

204. *To find the C. G. of a Material Straight Line.*

We may suppose the line to be made up of equal particles equidistant from its middle point. The C. G. of each pair of particles will be at the middle point of the line joining them (compare Art. 177); therefore the C. G. of the line will be at its middle point.

205. *To find the C. G. of a Parallelogram.*



The figure may be supposed to be made up of an indefinitely great number of material straight lines each parallel to *AB*.

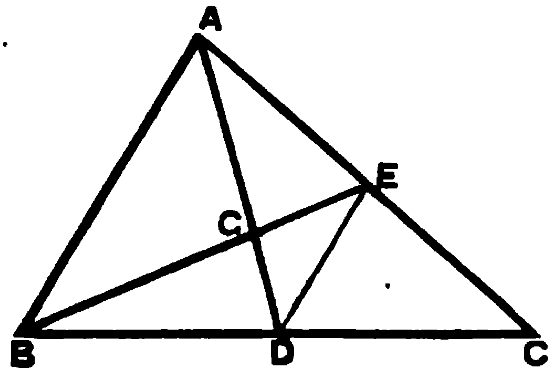
By Article 204, the C. G. of each is at its middle point.

The line *EF*, joining the middle points of *AB* and *DC*, will bisect every line parallel to *AB*.

∴ The C. G. of the figure is in *EF*.
Similarly it is in *HK*, which joins the middle
points of *BC* and *AD*.
∴ *G*, the point where *EF* and *HK* intersect,
is the C. G. of the parallelogram.

206. *To find the C. G. of a Triangular Lamina.*

We may suppose the triangle to be made up of an indefinitely great number of material lines parallel to the side *BC*.



Since the C. G. of each is at its middle point, and since any line from *A* to *D*, the middle point of *BC*, bisects every parallel to *BC*,
∴ the C. G. of the figure
is in *AD*,
the *median*[1] to the side *BC*.

---

[1] A *median* is the straight line drawn from any angle of a triangle to the middle point of the opposite side.

Similarly, the C. G. of the figure is in *BE*,
the median to the side *CA*;
$\therefore$ *G*, the point where the medians intersect,
is the C. G. of the figure.

To determine the position of G in the line *AD*.

Join *DE*. Then *DE* is parallel to *BA*, by Euc. vi. 2.

$\therefore$ the triangles *CDE*, *CBA* may be shown to be equiangular by applying Euc. i. 29.

$\therefore CD : DE = CB : BA$ (Euc. vi. 4.)

$\therefore$ Alt. $CD : CB = DE : BA$;

but $CD = \frac{1}{2} CB$;

$\therefore DE = \frac{1}{2} BA$.

Again, the triangles *DGE*, *AGB* are equiangular;

$\therefore DG : DE = GA : AB$;

$\therefore$ Alt. $DG : GA = DE : AB$;

but $DE = \frac{1}{2} BA$;

$\therefore DG = \frac{1}{2} GA$;

and therefore *DG* must be $\frac{1}{3} AD$.

To find the C. G. of any Triangle, it is sufficient either (1) to draw two medians, and their intersection will be the C. G.; or, (2) to draw a line from any angle to the middle point of the opposite side, and the C. G. will be in this line at one-third of its length from the side.

207. *To show that the C. G. of a triangle coincides with the C. G. of three equal heavy bodies at its angular points.*

Let three equal bodies be placed at *A*, *B*, *C*, and let the weight at each $= W$.

Then the C. G. of *W* at *B* and *W* at *C* will be at *D*, the middle point of *BC*.

We have now $2W$ at *D* and *W* at *A*.

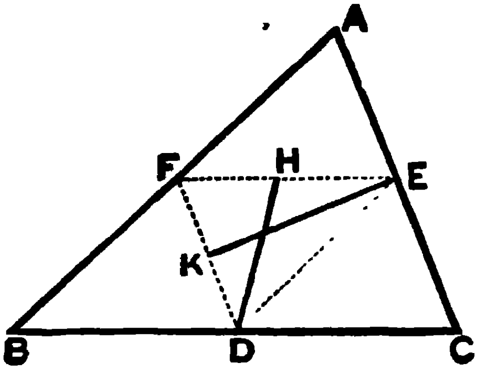
Their C. G. is at a point *O* (suppose) in *AD*, such that $AO : OD = 2W : W$; $\therefore AO = 2OD$; $\therefore DO = \frac{1}{3} DA$;

and therefore *O* coincides with *G*.

Conversely, if the weight of the triangle be W acting at G, we can replace this force by three vertical forces, each of which is equal to $\frac{W}{3}$, acting at A, B, and C respectively.

NOTE.—This device of supposing $\frac{1}{3}$ of its weight to act at each angle of a triangle often simplifies a Statical problem.

208. *To find the C. G. of the Perimeter of a Triangle*—A uniform wire bent into the form of a triangle will represent the perimeter. The weight of each side is proportional to its length, and the C. G. of each side is at its middle point.



We have therefore Weights $\equiv a, b, c$, situated at D, E, F, respectively.

Join FE, and divide it at H, so that $FH : HE = b : c$;

$\therefore$ the C. G. of b and c is at H (Art. 177);

$\therefore$ C. G. of a, b, c is in DH.

Now by EUC. vi. 2, 4, it may be shown that $DF = \frac{1}{2}AC = \frac{1}{2}b$,

and $DE = \frac{1}{2}BA = \frac{1}{2}c$;

$\therefore DF : DE = b : c$;

but $FH : HE = b : c$;

$\therefore FH : HE = DF : DE$;

$\therefore DH$ bisects the angle FDE (EUC. vi. 3)

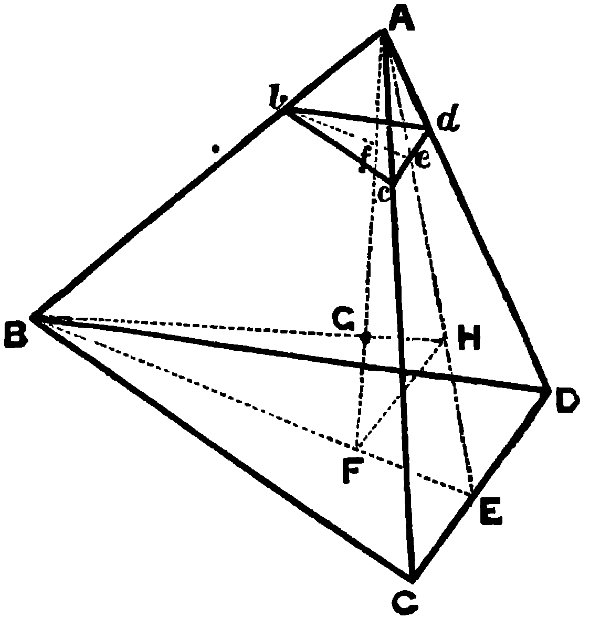
$\therefore$ C. G. of a, b, c lies in the line bisecting the angle FDE.

Similarly it may be shown that the C. G. lies in EK, which bisects the angle DEF.

$\therefore$ the C. G. is at the intersection of the lines which bisect the angles of the triangle DEF.

$\therefore$ the C. G. of the Perimeter of the triangle ABC *coincides with the centre of the circle inscribed in DEF.* (*See* EUC. iv. 4.)

209. *To find the C. G. of a Pyramid on a triangular base.*



Let BCD be the base, and A the vertex.

Bisect CD in E; join AE, BE; in EB take $EF=\frac{1}{3}EB$.

Then F is the C. G. of the base BCD. Join AF.

The Pyramid may be considered as made up of an indefinitely great number of planes parallel to the base BCD. Suppose bcd to be one of these planes.

Join AE, cutting cd in e, and join be.

By similar triangles (EUC. vi. 4) it may be shown that $ce=ed$.

Also, by similar triangles, $\frac{bf}{BF}=\frac{Af}{AF}=\frac{fe}{FE}$.

$\therefore$ Alternately $\frac{bf}{fe}=\frac{BF}{FE}$.

Now $BF=2FE$; $\therefore bf=2fe$;

and therefore f is the C. G. of the triangle bcd.

Hence the C. G. of the pyramid must be in AF, the line which joins the vertex with the C. G. of the face BCD.

Again, if H is the C. G. of the face ACD,

the C. G. of the pyramid is in BH;

$\therefore G$, the point of intersection of AF and BH,

is the C. G. of the pyramid.

To determine the position of G.

Join FH. Then, by Euc. vi. 2, FH is parallel to BA.

$$\text{Then } \frac{EF}{FH}=\frac{EB}{BA}; \text{ but } EF=\tfrac{1}{3}EB;$$

$$\therefore FH=\tfrac{1}{3}BA.$$

The triangles FGH, AGB are similar;

$$\therefore \frac{FG}{FH}=\frac{GA}{BA}; \text{ but } FH=\tfrac{1}{3}BA;$$

$$\therefore FG=\tfrac{1}{3}GA; \text{ and } \therefore FG=\tfrac{1}{4}FA.$$

Hence the C. G. of the pyramid is in the line joining the vertex with the C. G. of the base, at a distance of one-fourth of its length from the base.

210. *To find the C. G. of a Cone.*

If the base of a pyramid be a polygon, it may be shown, by the method of the last Article, that the C. G. of such a pyramid is in the line joining the vertex with the C. G. of the base, and at a distance from the base equal to one-fourth of this line.

A cone may be considered as a pyramid having as base a polygon with an infinite number of sides.

$\therefore$ the C. G. of a Cone is in the line joining the vertex with the centre of the base at a distance from the base equal to *one-fourth* of this line.

Note.—The C. G. of the *Surface of a cone* is in the line joining the vertex with the centre of the base at a distance from the base equal to *one-third* of this line. Consider the surface as made up of an infinite number of plane triangles, the sum of whose bases forms the circumference of the base of the cone and having their vertices at the vertex of the cone. The student will apply the result of Art. 206.

211. In Article 207 it was shown that the C. G. of a triangular lamina coincides with the C. G. of three equal weights at its angular points. This is a particular case of a wider Proposition; viz.: The C. G. of any *regular* figure, plane or solid, coincides (1) with the C. G. of its perimeter; and (2) with the C. G. of equal weights acting at its angular points.

Example i.—The C. G. of a body being given, and also the C. G. of a portion of it, find the C. G. of the remainder.

Let G be the C. G. of the body, and G_1 that of the part.

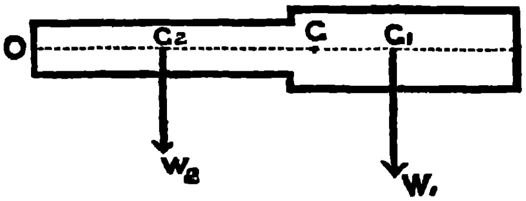
Suppose W to be the weight of the body, and W_1 the weight of the part. Then weight of the remainder $= W - W_1$.

Produce the line G_1G, and in G_1G produced take G_2 such that $G_1G : GG_2 = W - W_1 : W_1$.

Then G_2 is the C. G. of the remainder.

The value of GG_2 can be found by Article 177.

Example ii.—Two uniform cylinders of the same material are joined end to end, their axes being in the same line. Their lengths are 8 in. and 6 in., and their radii 2 in. and 3 in. respectively. Find their common C. G.



$W_1 = \pi(3)^2 \times 6 = 54\pi$;

$W_2 = \pi(2)^2 \times 8 = 32\pi$.

$G_1O = 8 + 3 = 11$ in.;

$G_2O = 4$ in.

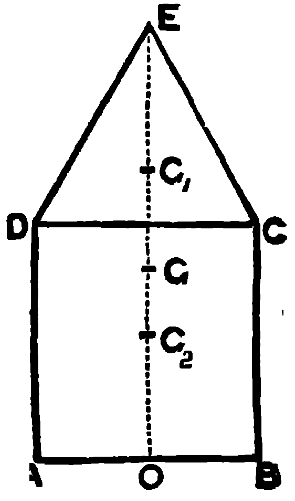
Take moments about O—

$$\therefore \quad (W_1 + W_2) \,.\, GO = W_1 \,.\, G_1O + W_2 \,.\, G_2O;$$

$$\therefore (54\pi + 32\pi) \,.\, GO = 54\pi \,.\, 11 + 32\pi \,.\, 4;$$

$$\therefore \quad GO = \frac{594 + 128}{86} = \frac{722}{86} = 8{\cdot}4 \text{ inches.}$$

Example iii.—$ABCD$ is a square whose side $=$ 10 feet. On CD as base an isosceles triangle CED is described externally of height 16 feet. Find the distance of their common C. G. from AB.



By the symmetry of the figure, the C. G. must be in the line EO, drawn perpendicular to AB.

$\Delta\ CED = 80$ sq. ft.; the square $= 100$ sq. ft.

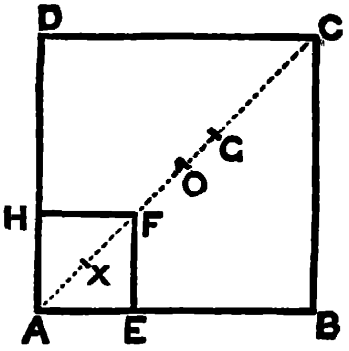
The weights may be supposed to be at right angles to EO.

Take moments about O—

$$\therefore (80 + 100)\ GO = 80\left(10 + \tfrac{16}{3}\right) + 100 \times 5.$$

From which equation, $GO = 9{\cdot}6$ ft.

Example iv.—From the corner of a square the side of which is 6 inches, a square whose side is 2 inches is cut out. Find the distance of the C. G. of the remainder from the centre of the square.



Area of $ABCD=36$;

Area of $AEFH=4$;

$\therefore$ Area of remainder $=32$.

The weights are supposed to act at right angles to AC.

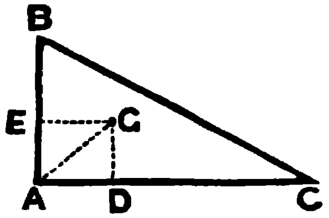
Take moments about the centre O—

$$\therefore 32 . GO=4 . XO;$$

$$\therefore GO=\frac{XO}{8}=\frac{AO-AX}{8}=\frac{\frac{1}{2}\times 6\sqrt{2}-\frac{1}{2}\times 2\sqrt{2}}{8}=\frac{\sqrt{2}}{4} \text{ inches};$$

$\therefore$ C. G. $=\frac{\sqrt{2}}{4}$ inches from the centre of the original square.

Example v.—Bodies weighing 10, 8, 6 are placed at the angles of a triangle whose sides are 25, 20, 15, the greatest weight at the greatest angle, and the least at the least. Find the distance of their C. G. from the 10.



The Δ is plainly right-angled. (Euc. i. 48.)

Let $BAC=90°$; $BA=15$, $AC=20$, $CB=25$.

The 10 is at A, 8 at B, 6 at C.

To find the distance of the C. G. from the line AC.

By Art. 196,

$$24\ GD=8\times 15; \quad \therefore GD=5$$

To find its distance from the line AB—

$$\therefore 24\ GE=6\times 20; \quad \therefore GE=5.$$

$$\text{Then, } AG^2=GD^2+DA^2=25+25=50;$$

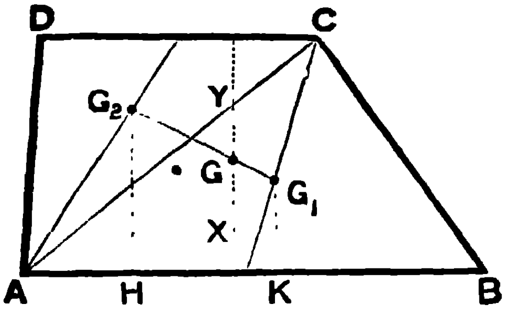
$$\therefore AG=\sqrt{50}=7{\cdot}07.$$

Example vi.—If a and b denote the parallel sides of a trapezium, and h the perpendicular distance between them, prove that if X and Y denote the distance of the C. G. of the figure from a and b respectively,

$$\text{then } X=\frac{h}{3}\times\frac{2b+a}{a+b}; \quad Y=\frac{h}{3}\times\frac{2a+b}{a+b}.$$

$$\Delta\ ABC=\frac{ah}{3}; \quad \Delta\, ACD=\frac{bh}{2}.$$

Then using the result of Art. 196 we get—



$$\therefore \quad \frac{ah}{2}\cdot G_1K+\frac{bh}{2}\cdot G_2H=\left(\frac{ah}{2}+\frac{bh}{2}\right)X;$$

$$\therefore \quad \frac{ah}{2}\cdot\frac{h}{3}+\frac{bh}{2}\cdot\frac{2h}{3}=\left(\frac{ah}{2}+\frac{bh}{2}\right)X;$$

$$\therefore \quad \frac{ah^2}{6}+\frac{2bh^2}{6}=\frac{h}{2}(a+b)X;$$

$$\therefore X=\frac{h^2}{6}(a+2b)\cdot\frac{2}{h(a+b)}=\frac{h}{3}\times\frac{2b+a}{a+b};$$

Then, $Y=h-X$, from which $Y=\frac{h}{3}\times\frac{2a+b}{a+b}$.

NOTE.—Another method is to bisect *AB* in *F*, and join *DF*, *CF*. Then, knowing the areas of the triangles *ADF*, *DFC*, and *FCB*, we can apply the result of Art. 196.

EXAMPLES—XXXI.

1. Show that the point of intersection of the medians is the C. G. of a plane triangle.

2. Find the C. G. of a parallelogram.

3. The equal sides of an isosceles triangle are 16 feet, and the base is 9 feet; find the distance of its C. G. from the sides.

4. Show how to determine the C. G. of a quadrilateral.

5. If a parallelogram be divided into four triangles by its diagonals, and the C. G.'s of these triangles be joined, show that these joining lines will form a parallelogram.

6. Show how, when the C. G. of the whole and also of a part of a body is known, the C. G. of the remainder may be found.

7. If a body is placed on a horizontal plane surface, how may we decide whether the body will stand or fall? Apply similar reasoning to the case of a body placed on an inclined plane.

8. A cubical block of wood is placed on a plane whose inclination is 60°; if prevented from sliding, will it have a tendency to roll?

9. Given the base and height of a triangle, construct it so that it will stand on a horizontal plane.

10. *ABCD* is a parallelogram having *ABC* 60°, and *BC* is 9 in. long; find the greatest possible length of *AB* that the figure may stand with *BC* placed on a horizontal plane.

11. If vertical triangles are placed on the same horizontal base *AB*, show that they will stand so long as their vertices are between two vertical lines distant $3AB$ from each other.

12. How many coins, in each of which the thickness is one-twentieth of the diameter, can stand in a cylindrical pile on an inclined plane of which the height is one-sixth of the base?

13. Show that the C. G. of a number of particles P, Q, R, . . . lying in a straight line OM at distances a, b, c, . . . respectively from O, is at a distance from $O = \dfrac{Pa+Qb+Rc+\ldots}{P+Q+R+\ldots}$.

14. By the last example, find the C. G. of four uniform straight rods, one of which is the diameter of a circle, and the others chords parallel to and on the same side of the diameter, subtending angles of 120°, 90°, 60° respectively at the centre.

Particles at the Angles of a Square.

15. Masses of 49, 73, 86 are placed at three angles *A*, *B*, *C* of a square whose side is *a*; find the distance of their C. G. from B.

16. Find the distance from 30 of C. G. of 15, 30, 45, placed at the angles of a square, if the side be 10 feet.

17. Masses of 17, 23, 28 are placed at *A*, *B*, *C*, the angles of a square *ABCD*; find the distance of their C. G. from *D*, if a side of the square = 8 ft.

18. Masses 3, 4, 5, 6 are placed at the corners of a square; find the distance of their C. G. from the 3, if a side = 2 feet.

19. Masses 3, 5, 7, 9 are placed at the angles of a square, whose side = 5 feet; find the distance of their C. G. from the 3.

20. Masses 5, 7, 10 are placed at three angles of a square whose side = 4 ft.; find the distance of their C. G. from the 5.

Particles at the Angles of a Triangle.

21. Three masses 3, 4, 5 are placed at the angles of an equilateral triangle whose side is 12 in.; find the distance of their C. G. from the least weight.

22. If masses 3, 4, 6 be placed at the angles of an equilateral triangle whose side is 10 in., find the distance of their C. G. from the least weight.

23. Find the distance from the 17 of the C. G. of three masses 10, 14, 17, placed at the angles of an equilateral triangle, each of whose sides is 3 ft. long.

24. Prove that the C. G. of a triangle coincides with that of three equal masses placed at its angular points.

25. Equal masses of 10 lbs. are placed at two of the angles of an equilateral triangle, the side of which is 5 ft., and a mass of 5 lbs. at the remaining angle; find the distance of the C. G. from the 5.

26. Masses 2, 3, 5 are placed at the angles of an equilateral triangle whose side is 6 ft.; find the distance of their C. G. from the greatest weight.

27. If the masses be 3, 4, 5, and the side$=6$ ft., find the distance of their C. G. from the 4.

28. Three masses 1, 2, 3 are placed at the angles of an equilateral triangle whose side$=a$; determine the distance of their C. G. from each of the angles in terms of a.

29. A triangle whose sides are 20, 16, 12 inches respectively has particles of equal mass at its angular points. Show that their C. G. is $6\frac{2}{3}$ in. from the greatest angle.

30. Three masses 33, 41, 56 are placed at the angles A, B, C of a right-angled isosceles triangle. Prove that the distance of their C. G. from the right angle B is equal to one-half of one of the equal sides.

When portions of a figure are removed.

31. $ABCD$ is a square whose middle point is E and whose side$=a$; if the triangle ECD be removed, find the C. G. of the remainder.

32. E and F are the middle points of the sides AB, AC of an equilateral triangle ABC; the portion AEF is removed; find the C. G. of the remainder.

33. The squares in Euclid ii. 4 about the diagonal are as $9:4$; find the C. G. of the greater gnomon.

34. A quarter of a triangle is cut off towards the vertex by a line drawn parallel to one side; find the C. G. of the remainder.

35. The *ninth* part of a triangle is cut off towards the vertex by a line parallel to the base. Find the C. G. of the remainder.

36. A portion of a triangle equal to $\frac{1}{n}$th of its area is cut off near one angle by a line parallel to the opposite side. Find the C. G. of the remainder.

37. From an isosceles triangle ABC, it is required to cut off a part CBD such that D may be the C. G. of the remainder.

38. The equilateral triangle ABC has each side $=4$ inches. From the corner A an equilateral triangle is cut off, having a side $=1$ inch; find the distance from A of the C. G. of the remainder.

39. If three equal triangles are cut off from a given triangle by lines drawn parallel to the sides, the C. G. of the remaining hexagon will coincide with that of the original triangle.

40. The vertex of a triangle is cut off by a line drawn parallel to the base, and the height of the figure is thus diminished by one-third; find the C. G. of the remainder.

41. Through the C. G. of a triangle ABC, a line DE is drawn parallel to the base BC; find the C. G. of the figure $DBCE$.

42. From a given square cut off a triangle having as base one side of the square, so that the C. G. of the remainder may be at the vertex of the triangle.

43. Out of an oblong figure is cut an isosceles triangle whose base coincides with one of the shorter sides; find where the vertex must be in order that the C. G. of the remaining figure may coincide with it.

44. The middle points of two adjacent sides of a square, each 5 feet in length, are joined, and the triangle so formed is cut off; find the C. G. of the remainder.

Figures made up of two or more figures.

45. An equilateral triangle is described upon one side of a square whose side $=16$ inches; find the distance of the C. G. of the figure so formed from the vertex of the triangle.

46. $ABCD$ is a square whose side $=2a$. On CD, as base, an isosceles triangle CED is described externally, whose altitude $=b$. Find the distance of the C. G. of the whole figure from AB.

47. A quadrilateral is formed by two isosceles triangles, whose heights are 12 inches and 9 inches respectively; find the distance of the C. G. of the figure from the common base of the triangles, the vertices being on different sides of it.

48. An isosceles triangle stands on a side of a square. If the C. G. of the figure be in the base of the triangle, compare the height of the triangle with the side of the square.

49. Squares are described externally on the sides and hypothenuse of a right-angled isosceles triangle; find the distance of the C. G. of the whole figure from the hypothenuse.

Perimeters.

50. Find the C. G. of the perimeter of a triangle, the sides of which are uniform material lines.

51. Three wires of equal weight, but unequal length, form the sides of a triangle; find their C. G.

52. Two circular rings, whose radii are r and r_1, are formed out of a rod of uniform thickness; find the distance of their C. G. from the point of external contact.

53. Two rings are formed of the same material, and placed in external contact, their diameters being 14 inches and 20 inches respectively; find the distance of their C. G. from the point of contact.

54. A uniform bar 8 feet long is bent so as to form four of the sides of a regular hexagon; find the distance of the C. G. from the centre of the circumscribing circle.

Solids.

55. Find the C. G. of a pyramid standing on a triangular base.

56. Show how the C. G. of a cone may be found.

57. How may the C. G. of a solid cylinder be found?

58. If two spherical bodies, whose radii are 6 inches and 9 inches, touch externally, find the distance of the common C. G. from the point of contact.

When bodies are suspended.

59. If a body be suspended from any point, what condition must be fulfilled for equilibrium?

60. If a heavy uniform lamina, in the shape of an equilateral triangle, be suspended from any of its angles, show that the opposite side is always horizontal.

61. A piece of wire is bent to form three sides of a rectangle, and is then hung up by one of its angles. If the sides containing that angle be equally inclined to the horizon, show that the ratio of the arms will be $\sqrt{3}-1:1$.

62. If a right-angled triangle be suspended from either of the points of trisection of the hypothenuse, show it will rest with one side horizontal.

63. A triangular lamina, when suspended from a point P in the side AB, rests with the side BC vertical; show that $AP=2BP$.

Miscellaneous.

64. Give examples of bodies in stable, unstable, and neutral equilibrium, and show how in each case the C. G. is affected by a slight displacement of the body.

65. If the C. G. of a triangle coincide with the C. G. of the inscribed circle, show that the triangle is equilateral.

66. Show that a body has one, and only one, Centre of Gravity.

67. Show that the locus of the C. G.'s of all right-angled triangles which can be described upon the same hypothenuse is a circle, and find the radius of that circle.

68. If a point be kept in equilibrium by forces represented in magnitude and direction by OP, OQ, OR, show that O is the C. G. of the triangle PQR.

69. Four 8-inch shells, each touching the other three, form a pyramid on a horizontal floor; find distance of their common C. G. from the floor.

70. Each of the sides AD, DG, GA, of an isosceles right-angled triangle ($A=90°$) is divided into three equal parts. At the successive points of division $A, B, C, \ldots I$, are placed particles whose weights are proportional to 1, 2, 3, . . . 9 respectively. Find the distance of their C. G. from the sides AD and AG.

71. If G be the C. G. of a triangle ABC, and from G and A two parallel lines be drawn to meet BC in E and D respectively, then $DE=\frac{1}{3}(BD \backsim DC)$.

72. Weights, P, Q, R, are placed at the vertices A, B, C respectively of a triangle, and their C. G. is at the centre of the circle described about the triangle. Prove that

$$\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}.$$

73. If G be the C. G. of a triangle ABC ($A = 90°$), and if AD, GE be drawn perpendicular to BC to meet it in D and E respectively, then shall $3BC.DE = AB^2 \backsim AC^2$.

74. If G be the C. G. of a triangle ABC, whose sides are 3, 4, 5, prove that $3(GA^2 + GB^2 + GC^2) = 50$.

75. A uniform square table (side $= a$ and weight $= W$) has masses 1, 3, 5, 7 lbs. suspended at its four corners in order, and is then found to balance on a point of support on its under surface. Show that the distances of this point from the sides connecting the masses (1, 7) and (7, 5) respectively are $\frac{a}{2}$ and $\frac{a}{2} \cdot \frac{W+8}{W+16}$.

76. From a square $ABCD$ a square $Abcd$ is cut out, Ab forming part of AB. If the gnomon DcB can just stand on a horizontal table with Bb as base, prove that $Ab = \frac{\sqrt{5}-1}{2}.AB$.

77. $ABCD$ is a trapezium, AB is parallel to CD, and $AB = \frac{1}{2}CD$. Show that the distance of the C. G. of the figure from AB is $1\frac{1}{4}$ times that from CD.

78. The sides of a 5-sided board $ABCDE$ are each $= a$, and the angles A and E are right angles. Prove that the distance of its C. G. from the side $AE = \frac{14+3\sqrt{3}}{26}.a$.

79. A book is placed on another similar book; show that three-fourths of the upper book will project beyond the table if, while the ends of the books are parallel to the edges of the table upon which they rest, they are so arranged that the upper projects as far as possible beyond the table.

80. A piece of wire is bent into the form of a rectangle, 3 feet by 1 foot. This rectangle can turn about the middle of one of its longer sides, and is placed in a position of unstable equilibrium. Two straight pieces of similar wire are then soldered to the ends of the vertical sides so as to be in a line with them and to render the equilibrium stable. What is the least length which these pieces must have?

81. If a and b denote the parallel sides of a trapezium, show that the C. G. of the figure lies on the line joining the points of bisection of a and b, and divides it in the ratio $2a+b : 2b+a$.

82. In the fig. of Art. 208, if the triangle DEF be removed, show that the C. G. of the remainder coincides with the C. G. of the original triangle ABC.

CHAPTER X.

CONDITIONS OF EQUILIBRIUM.

212. We shall treat only of forces whose lines of action are in the same plane.

When three forces which are not parallel act on a body and keep it in equilibrium, their lines of action must pass through the same point.

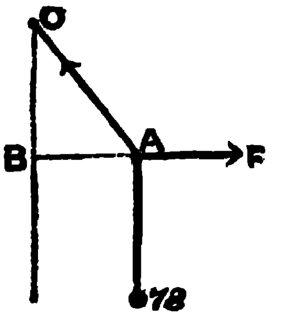
This may be proved as follows. The student is advised to draw figures to illustrate the reasoning.

Produce the lines of action of two of the forces to meet. These may be compounded into a single force. (Art. 160.) Because the forces are in equilibrium, therefore the third force must be equal and opposite to this new force, and must therefore act through the same point.

Note.—This fact alone is often sufficient to determine the position of equilibrium of a body.

In Statics it is usual to adopt the *weight of a pound* as the Unit of Force. (See Art. 118.)

Example i.—A body whose weight is 78 lbs. hangs vertically; what horizontal force applied at a point in the string will draw the upper part of the string aside through an angle of 30°?



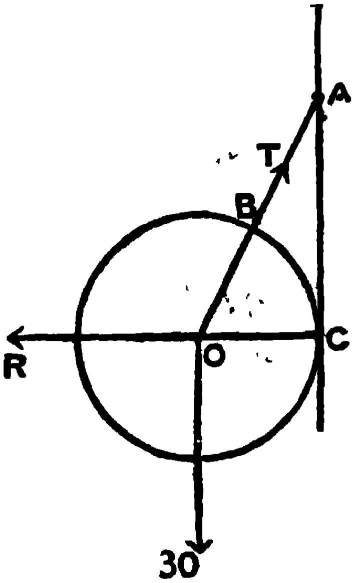
The point A is at rest under the three forces, T along the string, F horizontal, and 78 lbs. hanging vertically.

By Lami's Theorem (Art. 164)—

$$\frac{F}{78}=\frac{\sin 150^\circ}{\sin 120^\circ}=\frac{\sin 30^\circ}{\cos 30^\circ}=\tan 30^\circ;$$

$$\therefore F=78\times\frac{1}{\sqrt{3}}=26\sqrt{3} \text{ lbs. weight.}$$

Example ii.—A sphere 24 in. in diameter and weighing 30 lbs. has a string 12 in. long fastened to a point in its surface, the other end being fastened to a point in a smooth vertical wall; find the pressure against the wall, and the tension of the string.



The point O (the centre of the sphere) is at rest under the three forces; T, the tension along the string, the line of action of which passes through O, because the lines of action of R, the reaction of the wall, and of W, the weight of the sphere, pass through O.

The angle $AOC = \cos^{-1}(\frac{1}{2}) = 60°$.

By Lami's Theorem—

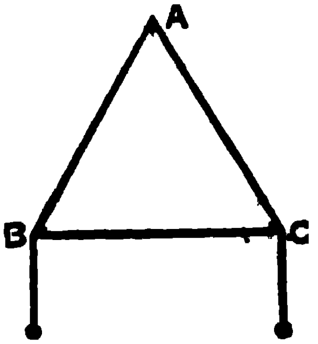
$$\frac{R}{30} = \frac{\cos 60°}{\sin 60°} = \cot 60° = \frac{1}{\sqrt{3}};$$

$$\therefore R = 10\sqrt{3} \text{ lbs. weight.}$$

$$\frac{T}{30} = \frac{\sin 90°}{\sin 60°} = \frac{2}{\sqrt{3}};$$

$$\therefore T = 20\sqrt{3} \text{ lbs. weight.}$$

Example iii.—Three smooth pegs, A, B, C, are driven into a vertical wall at the angles of an equilateral triangle, of which the base BC is horizontal. A string passing over the pegs supports two masses weighing 20 lbs. each at its extremities; find the pressures on the pegs.



The symmetry of the figure will show that the pressures on B and C must be equal.

The tension is the same throughout the string, and = 20 lbs. weight.

Let R_1 = the pressure on A.

Then R_1 and the two tensions along AB, AC are in equilibrium;

∴ the direction of R_1 bisects the angle BAC,

By Lami's Theorem, $\frac{R_1}{20} = \frac{\sin 60°}{\sin 150°} = \tan 60° = \sqrt{3}$.

$$\therefore R_1 = 20\sqrt{3} \text{ lbs. weight.}$$

Or, we might use the method of Article 160.

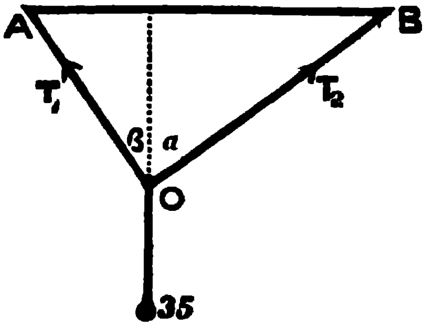
Let R_2 = the pressure on B.

$$R_2^2 = 20^2 + 20^2 + 2 \times 20 \times 20 \cos 150°.$$

From which $R_2 = 20\sqrt{2 - \sqrt{3}}$ lbs. weight.

The student may also find the values of R_1 and R_2 by resolving the forces vertically and horizontally; and also their values by Lami's Theorem.

Example iv.—Two strings, AO and BO, 3 ft. and 4 ft. long respectively, are tied together at O, and [illegible] fastened to two points A and B in the same horizontal line, 5 feet apart. A body whose mass is 35 lbs. is suspended from O; find the tensions in OA and OB.



By Euc. i. 48,

$$\because AB^2 = AO^2 + OB^2,$$

we know that the angle $AOB = 90°$.

The point O is at rest under the action of the two tensions T_1, T_2, and the weight acting vertically.

$$\text{Now } \beta = 90° - OAB = OBA,$$

$$\text{And } \alpha = 90° - OBA = OAB.$$

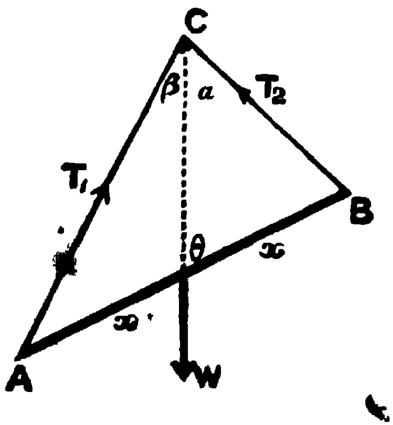
By Lami's Theorem, $\dfrac{T_1}{35} = \dfrac{\sin A}{\sin 90°} = \dfrac{4}{5}$;

$\therefore$ $T_1 = 28$ lbs. weight.

And $\dfrac{T_2}{35} = \dfrac{\sin B}{\sin 90°} = \dfrac{3}{5}$;

$\therefore$ $T_2 = 21$ lbs. weight.

Example v.—Two strings are fastened at one end to the same peg, and at their other ends to the extremities of a uniform rod. When the rod is at rest, show that the tensions in the strings are proportional to their lengths.



The weight of the rod acts at its middle point, and the direction of the weight passes through C. (Art. 198.)

Let the length of the rod be $2x$.

By Lami's Theorem, $\dfrac{T_1}{T_2} = \dfrac{\sin \alpha}{\sin \beta}$.

Now, $\dfrac{x}{BC} = \dfrac{\sin \alpha}{\sin \theta}$; and $\dfrac{x}{AC} = \dfrac{\sin \beta}{\sin (180 - \theta)}$.

$$\therefore \frac{\sin \alpha}{\sin \beta} = \frac{AC}{BC};$$

$\therefore$ $T_1 : T_2 = AC : BC$. (Q.E.D.)

EXAMPLES—XXXII.

1. A sphere 24 in. diameter, and weighing 80 lbs., has a cord a foot long fastened to a point in its surface; the other extremity being fastened to a point in a smooth vertical wall, find the pressure against the wall, and the tension of the cord.

2. If the sphere in Ex. 1 be 12 in. diameter, and weigh 25 lbs., and the string be 4 in. long, find the pressure against the wall, and the tension of the string.

3. A body weighing 100 lbs. is suspended by a rope passing over a fixed peg A; the end of the rope is made fast to the ground at a point B. If AB make an angle of 30° with the horizon, find the strain on the peg.

4. Two bodies weighing 60 lbs. each are suspended by a string passing freely over 3 pegs which form an isosceles triangle, the base of which is horizontal and the vertical angle is 90°; find the pressure on each peg.

5. Three pegs in a vertical wall are placed at the angles of an equilateral triangle, the two lower ones being in a horizontal line. A string passing over the three pegs supports a body weighing 5 lbs. at each end. Find the tension of the string, and the pressure on each peg.

6. If three forces be in equilibrium, show that their lines of action must be in the same plane.

7. A string of length $7a$ is fastened to the ends of a uniform rod, whose length is a, and passes over six smooth tacks placed so that the rod hangs in a horizontal position, and the whole forms a regular octagon; find the pressure on each of the tacks, and the tension of the string.

8. Two cords of equal length, each sustaining a body weighing 15 lbs., pass over two pulleys A and B, 3 ft. apart and in the same horizontal line, and the ends are fastened at C to a third weight; find this weight, if C be 4 ft. below the line AB.

9. A smooth ring, sustaining a body whose weight is 70 lbs., slides along a cord fastened at two points in the same horizontal line; find the tension of the cord, the parts of the string being inclined at right angles to each other.

10. If, in the last Example, the body weigh 90 lbs., and the parts of the string include an angle of 60°, find the tensions.

11. A cord PAQ is fastened to a point A, and is drawn in different directions by forces 3 and 7 in such a manner that the strain on $A = \sqrt{79}$. Find the angle PAQ.

12. A body weighing 15 lbs. is suspended by a string AB from a fixed point A; find what horizontal force at B will cause AB to make an angle of 45° with the vertical.

13. If the body in the last Example weigh 10 lbs., and the angle through which the upper part of the string is drawn aside be 30°, find the force.

14. A and B are two hooks, in the same horizontal line, to which the ends of a cord 15 feet long are attached. If a body weighing 130 lbs. hang from the middle of the cord, and the cord can bear the strain of 140 lbs. weight, find the greatest distance between A and B consistent with the safety of the cord.

15. A man weighing 11 st. lies in a hammock suspended by strings inclined at 30° and 45° respectively to the horizon; find the tensions in the ropes correctly to within $\frac{1}{2}$ lb. weight.

16. Two strings, at right angles, have their upper ends fixed at two points in the same horizontal line, and their lower ends fastened to a weight which hangs freely; show that their tensions are inversely as their lengths.

17. A drawbridge AB, whose length is 24 feet, and weighing a ton, is kept in a horizontal position by a chain 40 feet long made fast to B and to a point in a wall vertically over A; find the tension of the chain.

18. A uniform horizontal beam AB, movable about A and weighing 200 lbs., has a body weighing 800 lbs. suspended from B, and is kept from moving by a rope BC fastened to a point C. If ABC is 60°, find the tension of the rope.

19. A heavy uniform beam AB is movable in a vertical plane about A, and is kept inclined to the horizon at an angle of 60° by a rope fixed at B and C, C being a point on the ground such that $AC=AB$. Find the tension of the rope.

20. A body weighing 150 lbs. is supported by two strings, 6 and 8 feet long respectively, fastened to it, and attached at the other ends to points in the same horizontal line 10 feet apart; find the tension in each string.

21. Two bodies, each weighing 15 lbs., are connected by a string over a smooth peg; determine the tension in the string, and the pressure on the peg.

[illegible]. A and B are two points in a horizontal line. A body whose weight is W is attached to A and B by two weightless strings AC, BC, such that $AC=AB$, and the angle $BAC=36°$. If T_1 and T_2 be the tensions of BC and AC, show that $T_1-T_2=\frac{W}{2\cos 18^\circ}$.

23. A uniform heavy beam AB is movable about one end at A, and is suspended by a rope fixed to the end B, and to a point C in the horizontal line AC; find the sine of the angle ABC when $BAC=60°$, and the tension of the rope is equal to the weight of the beam.

24. A heavy uniform rod rests in a horizontal position with one end on a smooth table, and is suspended by a string attached at one-third of the length of the rod from the other end; find the *direction* and *tension* of the string.

25. A horizontal rod is suspended by two strings, each a yard long, passing over a smooth peg placed 1 foot vertically above the middle point of the rod. The ends of each string are attached respectively to one end and to the middle point of the rod. Show that the tension of each string is one-third of the weight of the rod.

26. A picture-frame is hung over a smooth peg; compare the tension of the string with the weight of the body, and find how the tension is affected by increasing the length of the string.

27. If a cord whose length is $2l$ be fastened at A and B, lying in the same horizontal line at a distance $2a$ apart, and if a smooth ring on the cord support a weight W, show that the tension of the cord $= Wl/2\sqrt{l^2-a^2}$.

28. A string, of which the ends are fixed at A and B, supports two weights W_1 and W_2, attached at points C and D in it; if AC and BD be produced to meet the lines of action of W_2 and W_1 in E and F respectively, show that $W_1 : W_2 = DE : CF$.

29. In Ex. 28 show that the strains on the points A and B are in the ratio $CE : DF$.

30. A weight of 50 lbs. is supported by two strings fastened to two points in the same level; the strings being at right angles to one another and of the same length, find the tension of each.

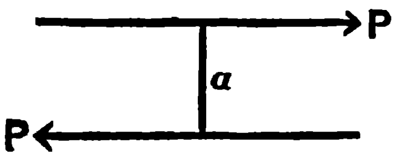
31. A ring is attached to one end of a string of length a, and the other end is fastened to a given point. Another string is fastened to a point in the same horizontal line as the first point, at a distance b from it, and this string, passing through the ring, supports a weight. If A be the angle which the first string makes with the horizon, prove that $b \cos 2A = a \cos A$.

32. Three forces F_1, F_2, F_3 are applied at the angles of a triangle ABC, and the line of action of F_3 bisects the angle C, while those of F_1 and F_2 make equal angles with the base AB. If the forces be in equilibrium, show that they are proportional to the sides opposite to the angles at which they act.

33. Two strings, bearing equal weights W_1 and W_1, and passing over two pulleys A and B, which lie in the same horizontal line at a distance apart $= 2a$, are fastened at a point C to a weight W. How far is the point C below AB when the system is in a position of equilibrium?

34. If PA, PB, PC represent three forces in one plane acting on a point P, show that their resultant passes through the intersection of the medians of the triangle ABC.

213. DEF.—A **Couple** is a system of two equal and parallel forces acting in opposite directions. (*See* Art. 179.)



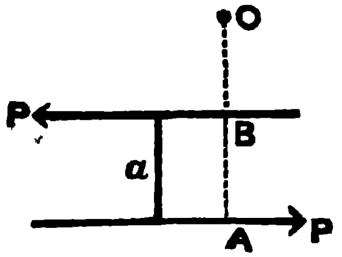
DEF.—The **Arm** of a couple is the perpendicular distance between the lines of action of the two forces.

DEF.—The **Moment** of a couple is the product of either force into the arm $= Pa$.

It is evident that the tendency of a couple which acts on a free body is to make the body rotate.

DEF.—Couples which tend to turn a body round in the same direction are said to be *like*. Couples which tend to turn a body round in opposite directions are said to be *unlike*.

214. *The Algebraical Sum of the moments of two forces of a Couple about any point in their plane is a constant.*



Let P, P be the forces of the couple;
$a =$ the arm;
O any point in the plane of the couple.
The Algebraical Sum of the Moments of the forces about O

$= P.OA - P.OB = P(OA - OB) = P.a$, a constant. (Q.E.D.)

And this constant can never be zero.

Similarly if O be within the forces of the Couple.

215. *When any number of forces act in the same plane on a body, and are not in equilibrium, they are equivalent either to a single force or to a Couple.*

Let a number of forces $F_1, F_2, F_3, \ldots . F_n$, act on a body.

Let $R_2 \equiv$ Resultant of F_1 and F_2.
" $R_3 \equiv$ " R_2 " F_3.
" $R_4 \equiv$ " R_3 " F_4.

Proceeding in this way, we eventually come to a point at which we are to find the resultant of R_{n-1} and F_n.

(1.) If these are equal and unlike parallel forces, we have a couple. (Art. 213.)

(2.) If they are unequal and parallel forces, they have a single resultant. (Arts. 177 and 178.)

(3.) If they are not parallel forces, they have a single resultant. (Art. 160.) (Q.E.D.)

COR.—If the system of forces be in equilibrium, then, since a Couple cannot produce equilibrium (Art. 214), we must have the last force F_n equal and opposite to the force R_{n-1}, the resultant of all the forces of the system except F_n.

Therefore, when a number of forces acting on a body are in equilibrium, *any one of them is equal and opposite to the resultant of all the others.*

216. *When any number of forces acting in the same plane on a body are in equilibrium, the Algebraical Sum of their moments about any point in the plane vanishes.*

The Moment of the resultant is equal to the Algebraical Sum of the Moments of the forces. (Art. 191.)

But the resultant being zero, its Moment vanishes.

∴ the Algebraical Sum of the Moments of all the forces vanishes. (Q.E.D.)

To find the Conditions of Equilibrium if a number of forces act on a body and in the same plane.

FIRST SET OF CONDITIONS.

217. The forces will be in equilibrium *if the Algebraical Sum of the moments of the forces vanishes round any three points* O_1, O_2, O_3, *in the plane, provided that these three points are not in the same straight line.*

Suppose the system not to be in equilibrium; then it must reduce to a couple or to a single resultant. (Art. 215.)

Take three points O_1, O_2, O_3.

By *data*, the sum of the Moments about O_1 vanishes.

$\therefore$ the system cannot reduce to a couple. (Art. 214.)

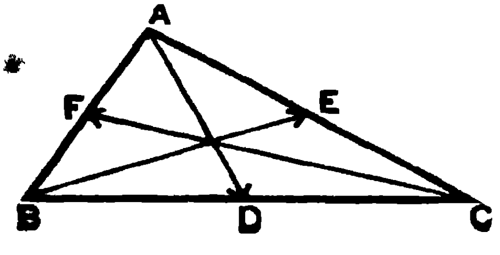
$\therefore$ O_1 is a point in the line of action of their resultant. (Art. 193.)

Similarly O_2 and O_3 are points in the line of action of their resultant.

$\therefore$ O_1, O_2, O_3, are in the same straight line, which is contrary to the hypothesis.

Therefore the system must be in equilibrium.

Example.—Three forces are fully represented by the three medians of a triangle; show that they are in equilibrium.



Take moments about A.

The moment of AD about $A = 0$ (Art. 185).

The moment of BE about A
$= 2 \triangle BAE = \triangle ABC.$

The moment of CF about A
$= -2 \triangle CAF = -\triangle ABC$ (Art. 183).

$\therefore$ Sum of the moments about A vanishes.

Similarly, sum of the moments about B vanishes.

" " " C "

$\therefore$ the forces are in equilibrium. (Q.E.D.)

SECOND SET OF CONDITIONS.

218. The forces will be in equilibrium (1) *If the Algebraical*

Sums of the components of the forces in any two directions vanish separately; (2) *If the Algebraical Sum of the moments of all the forces about any point in the plane vanishes.*

NOTE.—It is generally more convenient, but not *necessary*, to choose the directions at right angles to each other. (Ex. xxvii. 39.)

By Article 215, if they are not in equilibrium, the system reduces to a single force, or to a couple.

Now by Article 170, if the Algebraical Sum of the forces resolved along two lines at right angles vanishes, the resultant vanishes.

.·. the system cannot reduce to a single force.

Again, by Article 214, if the system reduces to a couple, then the moment of this couple about a point is a constant which is not zero, and therefore the Algebraical Sum of all the forces about the point cannot vanish.

.·. the system cannot reduce to a couple.

.·. the system must be in equilibrium.

The latter set of conditions is the more useful in the practical solution of problems. The conditions are easily remembered if stated as follows:—

(1.) *The sum of the horizontal components* = 0.
(2.) *The sum of the vertical components* = 0.
(3.) *The sum of the moments about any point in the plane* = 0.

219. We now proceed to work some typical examples in which forces, not acting on a *particle*, are in equilibrium. But before doing so, the student ought to notice the following points connected with Statical Problems.

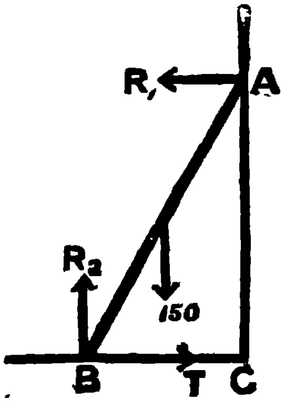
(i.) The tension of a string stretched over a smooth peg is the same on both sides of the peg.

(ii.) The direction of the resultant of two equal forces acting on a particle bisects the angle between them.

(iii.) The reaction from a smooth surface is always at right angles to the surface.

(iv.) If a rod press against a smooth plane, the mutual pressure is at right angles to the *plane*; if a smooth rod rest on a peg, or in a ring, the mutual pressure is at right angles to the *rod*.

Example i.—A uniform beam *AB*, 25 feet long, rests with one end on a smooth horizontal plane, and the other against a smooth vertical wall. If a string 15 feet long connect the lower end with the foot of the wall, find (1) the tension of the string; (2) the pressure at the top; (3) the pressure at the bottom; having given that the weight of the beam is 150 lbs.



Let $R_1\,R_2 \equiv$ the reactions at *A* and *B* respectively.

$T \equiv$ the tension in the string *BC*.

Equating *horizontal* forces we have—

$$T = R_1 \dots\dots\dots\dots \text{(1.)}$$

Equating *vertical* forces, we have—

$$R_2 = 150 \dots\dots\dots\dots \text{(2.)}$$

Equating *moments* about *B*, we have—

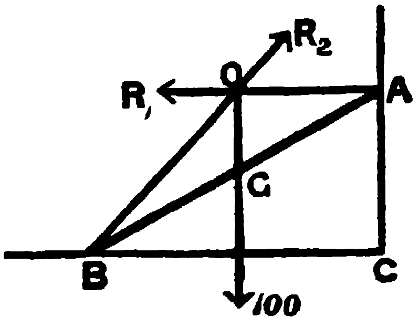
$$R_1 \times 25 \sin ABC = 150 \times \tfrac{25}{2} \cos ABC \dots \text{(3.)}$$

Now, $\cos ABC = \frac{15}{25} = \frac{3}{5}$; $\therefore \sin ABC = \frac{4}{5}$.

Solving the three equations, (1), (2), (3), we get—

$T = 56\frac{1}{4}$ lbs. weight; $R_1 = 56\frac{1}{4}$ lbs. weight; $R_2 = 150$ lbs. weight.

Example ii.—A uniform ladder, 36 feet long, rests with one end *against* a smooth wall, and the lower end is prevented from slipping by a peg. If the inclination of the ladder to the horizon be 30°, find the pressure on the wall, and at the peg, the ladder weighing 100 lbs.



Let R_1 and $R_2 \equiv$ the reactions at the wall and peg.

R_1 will be at right angles to the *wall*.

Let the lines of action R_1 and of the weight meet in *O*.

$\therefore$ the line of action of R_2 must also pass through *O*. (Art. 212.)

Equate horizontal components—

$$\therefore R_1 = R_2 \cos OBC \dots\dots\dots\dots \text{(1.)}$$

Equate vertical components—

$$\therefore R_2 \sin OBC = 100 \dots\dots\dots\dots \text{(2.)}$$

Equate moments about B—

$\therefore R_1 \times 36 \sin 30° = 100 \times 18 \cos 30° \;.\;.\;.\;.\;.\;.\; (3.)$

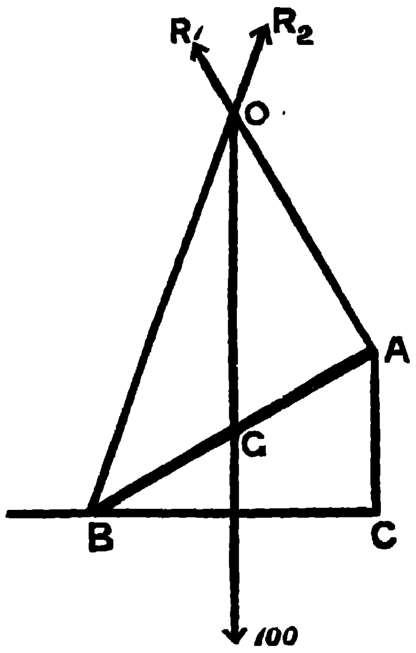
From (3), $R_1 = 50\sqrt{3}$ lbs. weight.

From (1) and (2), $\tan OBC = \frac{2}{\sqrt{3}}$; $\therefore \sin OBC = \frac{2}{\sqrt{7}}$

Hence $R_2 = 50\sqrt{7}$ lbs. weight. (*See* Ex. xxxiv. 16.)

Example iii.—If the ladder in Ex. ii. rested with its upper end *on* the wall, find the pressures on the wall and the ground.

In this case R_1 is at right angles to the *ladder*. Then, if the directions of R_1, and of the weight pass through O, the direction of R_2, also passes through O.



Equate moments about B—

$\therefore R_1 \times 36 = 100 \times 18 \cos 30° \;.\;.\;.\; (1)$

From which $R_1 = 25\sqrt{3}$ lbs. weight.

Equate vertical components—

$\therefore R_1 \cos 30° + R_2 \sin OBC = 100$ (2.)

Equate horizontal components—

$R_1 \sin 30° = R_2 \cos OBC \;.\;.\;.\; (3.)$

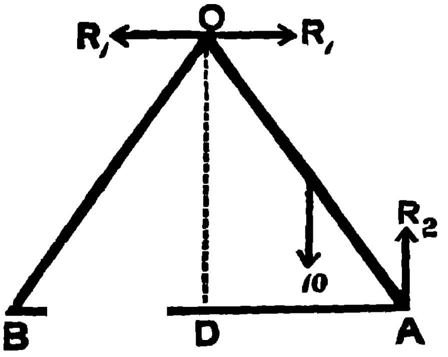
From the equations (2) and (3)—

$\tan OBC = \frac{5}{\sqrt{3}}$;

and $R_2 = 25\sqrt{7}$ lbs. weight.

(*See* Examples xxxiv. 15.)

Example iv.—Two equal rods OA, OB, each $16\frac{1}{4}$ feet long and weighing 10 lbs., are connected at O, and their other ends are placed on a smooth horizontal plane, A, B, O being in the same vertical plane. If a string $20\frac{2}{5}$ feet long connect A and B, find (1) the pressure at O, (2) the pressure at A and B, (3) the tension in the string.



The figure being symmetrical with regard to the vertical line OD, we may treat the question as in Ex. 1.

$OD^2 = OA^2 - AD^2 = (16\frac{1}{4})^2 - (10\frac{1}{5})^2$

From which $OD = \frac{253}{20}$.

Let $T \equiv$ the tension in the string AB.

Equate vertical components—

$\therefore R_2 = 10 \;.\;.\;.\;.\;.\;.\; (1.)$

Equate horizontal components—

$\therefore T = R_1 \;.\;.\;.\;.\;.\; (2.)$

Equate moments about A—

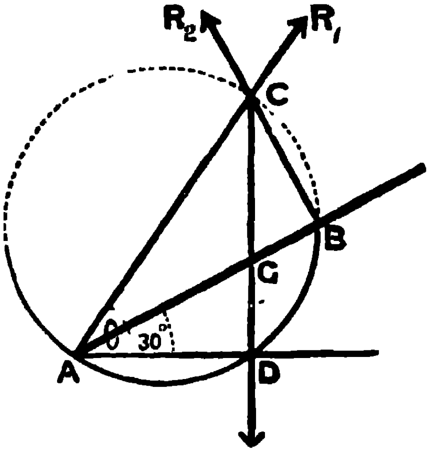
$$\therefore R_1 \times 16\tfrac{1}{4} \sin OAD = 10 \times 8\tfrac{1}{8} \cos OAD \quad . \; . \; . \; . \; . \quad (3.)$$

From (3), $R_1 = 4\frac{8}{263}$ lbs. weight,

$\therefore$ From (2), $T = 4\frac{8}{263}$ lbs. weight.

Thus R_1, R_2, and T have been determined.

Example v.—When a uniform rod 16 feet long and weighing 40 lbs. rests in a hemispherical bowl, the axis of which is vertical, the rod making an angle of 30° with the horizon, and extending 4 feet beyond the bowl, find the pressures at the foot of the rod, and at the rim.



R_2, the reaction at B, is at right angles to AB.

R_1, the reaction at A, is normal to the surface.

$\therefore$ C, the point where the lines of action of R_1 and R_2 meet, is on the circumference of the circle which represents the section of the completed sphere.

The line of action of the weight must also pass through C. (Art. 212.)

The point D, where the line of action of the weight cuts the horizontal line must be also on the circle.

$AG = 8$ feet; $GB = 4$ feet.

Equate moments about A—

$$\therefore R_2 \times 12 = 40 \times 8 \cos 30° \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.)$$

From this equation $R_2 = \dfrac{40\sqrt{3}}{3}$ lbs. weight.

Equate vertical components—

$$\therefore R_2 \cos 30° + R_1 \sin \theta = 40 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

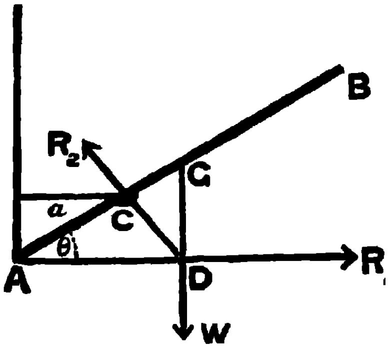
Equate horizontal components—

$$\therefore R_2 \sin 30° = R_1 \cos \theta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.)$$

From (2) and (3) we shall find $\theta = 60°$; $R_1 = \dfrac{40\sqrt{3}}{3}$ lbs. weight.

(*See* Examples xxxiii. 25 for the general case.)

Example vi.—A uniform rod whose length is $2l$ rests with one end against a smooth vertical wall and is placed across a horizontal rail distant a from the wall; find the position of equilibrium, and the pressures on the wall and rail.



R_1 and R_2, the reactions at the wall and rail, are at right angles to the wall and rod respectively.

The lines of action of R_1, R_2, and W must pass through the same point D. (Art. 212.)

Equate vertical components—

$$\therefore R_2 \cos\theta = W \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Equate horizontal components—

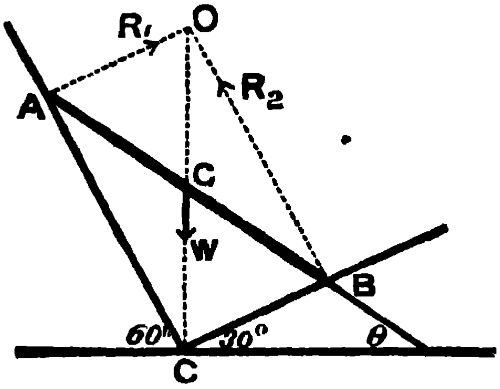
$$\therefore R_2 \sin\theta = R_1 \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Equate moments about A—

$$\therefore R_2\, a \sec\theta = Wl\cos\theta \quad . \quad . \quad . \quad (3.)$$

From (1) and (3) we may find $\cos^3\theta = a/l$, and then R_1 and R_2 may be shown to be equal to $\frac{W\sqrt{l^{\frac{2}{3}} - a^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$, and $W\left(\frac{l}{a}\right)^{\frac{1}{3}}$.

Example vii.—A uniform rod, 10 feet long, is placed on two smooth planes whose inclinations to the horizon are 30° and 60° respectively: find the pressure on each plane, and the inclination of the rod to the horizon when in equilibrium, the weight of the rod being 40 lbs.



R_1 and R_2, the reactions at A and B, are at right angles to the planes. Since there is equilibrium, the direction of the weight passes through O, their point of intersection. (Art. 212.)

The figure $OACB$ is a rectangle;

$$\therefore AOC = OCB = 60°;\ BOC = OCA = 30°.$$

By Lami's Theorem $\frac{R_1}{40} = \frac{\sin 30°}{\sin 90°}$; from which $R_1 = 20$ lbs. weight.

Also $\frac{R_2}{40} = \frac{\sin 60°}{\sin 90°}$; from which $R_2 = 20\sqrt{3}$ lbs. weight.

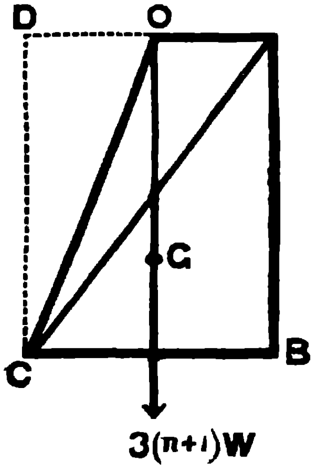
To find θ; $\angle\ GBC = \angle\ GCB = 60°$.

Then $\theta = 60° - 30° = 30°$.

The *general* case of this problem is given as an exercise for the student in Examples xxxiv. 17.

NOTE.—The same method of solution may be used when a heavy sphere rests between two inclined planes. (Examples xxxiii. 24.)

Example viii.—From a given rectangle $ABCD$ cut off a triangle CDO (O being in AD), so that when the figure $ABCO$ is suspended from O the sides AO, BC may be horizontal.



Let a = side AD, and let $AO = na$;

Then $OD = (1-n)a$.

The $\frac{\triangle ACO}{\triangle ACD} = \frac{AO}{AD} = \frac{na}{a} = n$.

Let weight of $\triangle ABC = 3W$.

Then weight of $\triangle ACO = 3nW$.

By Art. 207 we may consider the weight of a triangle divided into three equal weights acting at the angles. Adopting this method we have acting—

at A, a weight $= W + nW = (n+1)W$; at B, a weight $= W$;
at C, ,, $= W + nW = (n+1)W$; at O, ,, $= nW$.

Taking moments about O we have—

(Weight at C) × DO = (Weight at A + Weight at B) × AO;

$\therefore\ (n+1)W \times (1-n)a = (\overline{n+1}\,W + W) \times na$; $\therefore\ 1 - n^2 = n^2 + 2n$.

$2n^2 + 2n - 1 = 0$; $\therefore$ from this equation $n = \frac{\sqrt{3}-1}{2}$.

But $\frac{AO}{AD} = n$; $\therefore\ AO : AD = \sqrt{3} - 1 : 2$.

NOTE.—This method of distributing the weight of a triangle equally at the angular points deserves particular attention.

EXAMPLES—XXXIII.

1. A uniform rod AB is suspended with its end in contact with a smooth vertical wall AC by a string CE; if $AE = \frac{1}{4}AB$, show that CB will be horizontal.

2. A uniform beam (length $= l$ and weight $= W$) rests at a given inclination θ to the horizon, with its ends in contact with a smooth vertical wall and a smooth horizontal plane, and is prevented from slipping by a string attached to its lower end and the foot of the wall; find the tension of the string and the pressure on each plane.

3. In Ex. 2, find the least value of θ when the string can just support a weight equal to 4 times that of the ladder.

4. A number of forces act in one plane on a particle. In order that there may be equilibrium the sums of the resolved forces in any two directions in the planes of the forces at right angles to one another must be severally zero; prove this.

5. If in Ex. 4 the forces act in one plane but their lines of action do not pass through the same point, what further condition is necessary in order that they may be in equilibrium?

6. A rod without weight, 5 inches long, has a string 7 inches long attached at both ends. The string is hung over a smooth peg, and weights 3 and 4 hang at the ends of the rod; show that there will be equilibrium when the rod and the parts of the string form a right-angled triangle.

7. An equilateral triangular lamina is suspended by a string fastened at two of its angles and passing over a smooth peg; show that if one side be vertical the length of the string used must be double the height of the triangle.

8. A uniform rod is supported by two strings inclined at given angles to the horizon; find their tensions and the inclination of the rod.

9. A spherical shot weighing 60 lbs. rests between two planes which are inclined at angles of 30° and 60° to the horizon; find the pressure on each plane.

10. A uniform rod, 13 inches long and weighing 26 oz., is suspended from a point by two strings 5 inches and 12 inches long respectively attached to its ends; find the position of equilibrium, and the tensions of the strings.

11. A body weighing 1 lb. slides on an endless string 32 inches long, which is put over two smooth pegs in the same horizontal line one foot apart; find the position of equilibrium, and the tension in the string.

12. A rectangular board *ABCD*, whose weight is *W*, and whose sides *AB*, *BC* are 15 inches and 5 inches respectively, is suspended by two vertical strings at *A* and *C* from two points 15 inches apart in a horizontal line; if the second string be 5 inches longer than the first, and a weight *W* be suspended from the point *B*, find the tensions of the strings.

13. If three forces acting on a body be in equilibrium, show that they must lie in the same plane, and that their lines of action must either be parallel or pass through one point.

14. A uniform beam, 32 feet long and weighing 200 lbs., rests with one end on a smooth horizontal plane, and the other against a smooth vertical wall. If a string 16 feet long connect the lower end with the foot of the wall, find (1) the tension of the string, (2) the pressure against the wall, (3) the pressure on the plane.

15. A uniform ladder, 40 feet long and weighing 180 lbs., rests with one end against a smooth wall, and is prevented from slipping by a peg in the ground; find the pressures against the wall and at the ground, if the inclination of the ladder to the horizon be 60°.

16. A lever AC, whose weight may be neglected, free to turn about the end C, has weights W and W_1 suspended from A and its middle point B respectively, and is kept at rest by a weight P acting at A by means of a string which passes over a peg D. If CD be horizontal and equal to AC, determine the position of equilibrium below horizon.

17. Two uniform beams, each 24 feet long and weighing 110 lbs., joined at one end, rest with their other ends fixed to the top of two vertical walls of the same height and 36 feet apart; find the horizontal thrust tending to overturn each wall.

18. A uniform beam, 12 feet long and weighing 50 lbs., rests with one end A at the bottom of a vertical wall, and a point C in the beam 10 feet from A is connected by a horizontal string CD with a point D in the wall 8 feet above A; find (1) the tension of the string, and (2) the pressure against the wall.

19. A spherical shot weighing W lbs. rests between a smooth vertical wall and a smooth plane, the inclination of the latter to the horizon being 45°; find the pressures on the wall and the plane.

20. A horizontal rod is supported by two strings, each 1 yard long, passing over a small smooth peg placed 1 foot vertically above the middle point of the rod. The ends of each string are attached respectively to one end and to the middle point of the rod. Show that the tension of each string is one-third weight of the rod.

21. A uniform beam, weight W, rests with one end on a smooth plane inclined at 15° to the horizon, and the other against a smooth vertical wall. Find the pressures on the wall and the plane, and the inclination of the beam to the horizon.

22. A uniform heavy beam AB rests against a smooth horizontal plane CA and a smooth vertical wall CB, the lower extremity A being attached to a string which passes over a smooth pulley at C and sustains a given weight P. Find the position of equilibrium and the pressures on the plane and the wall.

23. Prove that if a uniform rod be suspended by a string fastened to its ends and passing over a smooth peg, it will be in equilibrium only if the rod be vertical or horizontal.

24. A sphere, whose weight is W, rests on two inclined planes, the angles of inclination being i and i_1; find the pressure on each plane.

25. Find the position of equilibrium of a uniform beam in a hemispherical bowl, when part of the beam projects beyond the bowl; m and n being the segments of the part inside made by the C. G., W the weight of the beam, and θ the inclination of the rod to the horizon. Find also the pressures at the lower end, and at the rim.

26. A trap door, weight W, turning on a hinge, is supported by a weight P hanging at the end of a string which passes over a pulley in the same horizontal plane as the hinge and at a distance equal to the length of the trap door. If θ be the angle below the horizontal at which the door is inclined when in equilibrium, show that

$$2P\cos\tfrac{1}{2}\theta = W\cos\theta.$$

MISCELLANEOUS EXAMPLES—XXXIV.

1. Two forces, each $= P$, neutralise each other when connected by a string over a fixed smooth peg. If the angle between the two parts of the cord $= \theta$, show that the strain on the peg $= 2P\cos\tfrac{1}{2}\theta$.

2. If three forces proportional to the sides of a triangle act along the sides, to what is the system equivalent?

3. Two masses M_1 and M_2, are placed on Ox at known distances from O; M_2 is then moved a known distance a along Ox; how far has the C. G. of the two masses been moved?

4. If the C. G. of a quadrilateral coincide with one of its angular points, show that the distances of this point and of the opposite angular point from the straight line joining the other two are in the ratio of 1 : 2.

5. A small ring of given weight rests on the arc of a smooth circular hoop which is fixed in a vertical plane, being attached to the highest point by a string whose length is equal to the radius of the hoop and resting on the hoop; find the tension of the string, and the pressure on the hoop.

6. Two uniform rods of the same material, 5 in. and 3 in. long, are connected at one end A so as to be at right angles, and are suspended by a string so that the other ends are in the same horizontal plane. Show that the string is fastened to a point in the longer rod at 1·225 in. from A.

7. If through a fixed point there be drawn three straight lines representing three forces in equilibrium, any one of them will, if produced, bisect the line joining the extremities of the other two.

8. Prove that if three points can be found (not in the same straight line) in the plane of a system of forces, such that the sum of the moments of the forces about each of them is zero, then the system is in equilibrium.

9. Two particles, of weight W_1 and W_2 respectively, are connected by a string which lies on a smooth vertical circle; show that if the part of the string in contact with the circle subtend an angle of 90° at the centre, the inclination to the horizon of the chord of that arc in the position of equilibrium $= \tan^{-1}\left(\frac{W_1 - W_2}{W_1 + W_2}\right)$.

10. Divide a body W into three parts so that when placed at A, B, C, the three angles of a triangle, their C. G. shall coincide with the C. G. of the wire forming its sides, the lengths of the sides being 4, 5, 6 inches respectively.

11. Find the position and magnitude of the resultant of two unlike parallel forces acting on a body; and prove that the algebraical sum of their moments about any point in their plane is equal to the moment of their resultant about that point.

12. Forces in equilibrium act in the lines joining the centre of the inscribed circle with the angles of a triangle; express their ratios in terms of the sides of the triangle.

13. A heavy rectangular lamina $ABCD$, weight W, and movable about a horizontal axis at A, is kept in equilibrium with AD vertical by the tension of a string acting along DB; find the pressure on A, and the angle between its direction and that of the string, having given that $AB = 2BC$.

14. Two uniform beams CA, CB, joined at C above A and B, rest with their ends on the tops of two walls, AD, BE, of the same height. If W be the weight, and α the inclination of each rod to the horizon, find the horizontal thrust on each wall.

15. A ladder, whose C. G. divides it into two segments, m and n, and whose weight is W, rests with one end on the *top* of a wall, and is prevented from slipping by a peg driven into the ground at the lower end. If α be the inclination of the ladder to the horizon, find the pressures at the base and on the wall.

16. If the ladder in Ex. 15 rested *against* the wall, find the pressures.

17. If a beam, whose C. G. divides it into two segments, m and n, rest on two planes whose inclinations are i and i_1, find the angle θ which it makes with the horizon in its position of equilibrium.

18. If the beam in Ex. 17 be suspended at each extremity by cords passing over two pulleys placed in the same horizontal plane and carrying weights P and Q at their other ends, find the angle which the beam makes with the horizon in its position of equilibrium.

19. Two uniform beams, connected at a given angle θ, turn about a horizontal axis at their point of connection; find the position of equilibrium which they take up by their own weight.

20. From a cone whose height is 18 in. a similar cone 6 in. high is cut off by a plane parallel to the base; what is the height of the C. G. of the frustum from the base?

21. Two equal masses (each $= W_1$) are connected by a string, which passes over two smooth pegs A, B, in the same horizontal plane, and supports a mass W_2, which hangs from a smooth weightless ring through which the string passes; find the depth of the ring below AB.

22. If two forces mOA and nOB act on a particle, show that their resultant is $(m+n)OG$, where G is a point in AB such that $AG : BG = n : m$.

23. If two spheres whose radii are a and b respectively touch internally, find the distance from the point of contact of the C. G. of the solid figure contained between the surfaces.

24. Two spheres whose radii are a and b rest between two smooth planes whose inclinations are i and i_1; at what angle is the line joining their centres inclined to the horizon?

25. Two cones have the same base, and their vertices lie towards the same parts; find the distance from their common base of the C. G. of the solid figure contained between their two surfaces.

26. Find the distance of the C. G. of the frustum of a cone from the base, a and b being the radii of the two ends, and h the height of the frustum.

27. Forces $k \cos \frac{1}{2}A$, $k \cos \frac{1}{2}B$ act along OB and OA, O being the centre of the circle inscribed in the triangle ABC. If the line of action of the resultant of these two forces and a force P along OC passes through the middle point of AB, find P.

28. Two forces $k \cos A$, $k \cos B$ act along the sides of the plane triangle ABC. Find their resultant, and show that its line of action divides the angle C into the two angles $\frac{1}{2}(C+B-A)$ and $\frac{1}{2}(C+A-B)$.

29. A heavy triangle ABC is suspended successively from the angles A and B, and the two positions of any side are found to be at right angles to each other. Prove that $5c^2=a^2+b^2$.

30. Three forces act at the middle points of the sides of a plane triangle, and at right angles to them; if they are proportional to the sides at which they act, show that they are in equilibrium, when all are directed either outwards or inwards.

31. If three forces in equilibrium act along the medians of a triangle, show that they must be proportional to those medians.

32. A string 9 feet long has one end attached to the extremity of a smooth uniform rod 2 feet long, and the other end carries a ring without weight which slides upon the rod. The rod is suspended by the string from a smooth peg. Show that θ, the angle which the rod makes with the horizon, is given by the equation

$$9 \tan^3\theta + 9 \tan\theta = 2.$$

33. A sphere, of weight W, rests on two similar smooth wedges which are placed on a smooth table, and are prevented from sliding by a horizontal string connecting them. Find the tension of the string.

34. To one end A of a uniform heavy rod CA, which can turn freely about a hinge at C, is attached a string which passes over a smooth pulley P (the distance CP being horizontal and equal to CA), and supports a heavy particle whose weight is half that of the rod. Show that the rod can rest at an angle of 30° to the vertical, and determine the magnitude and direction of the action at the hinge.

35. Two posts, one of which is $a(\sqrt{3}-1)$ feet higher than the other, stand at a horizontal distance $a(\sqrt{3}+1)$ feet apart. A body whose weight is W hangs by two strings, of length $2a\sqrt{2}$ feet, attached each to the top of one of the posts. Find the tensions of the strings.

36. A sphere of radius a and weight W is supported on a smooth plane inclined at an angle θ to the horizon by a string of length b whose ends are fastened respectively to a point on the surface of the sphere and a fixed point in the plane. Find the tension of the string.

37. A smooth sphere, of radius 1 foot and weight 4 lbs., is kept at rest on a smooth plane inclined at an angle θ to the horizon by a smooth uniform beam hinged to the inclined plane, and inclined to it at an angle β, and resting on the sphere. The length of the beam being 8 feet, and its weight 10 lbs., and $\theta+\beta$ being equal to 45°, show that

$$\sin\theta = 10\sqrt{2}\sin^2 \tfrac{1}{2}\beta.$$

38. ABC is a triangle, X, Y, Z are points in the sides BC, CA, AB respectively, and $BX : XC = CY : YA = AZ : ZB$. Show that three forces represented in magnitude and direction by AX, BY, CZ acting at a point are in equilibrium.

39. A heavy rod (length $=2a$), whose particles are disposed after the manner of a cone or pyramid of very small base, is put into a smooth sphere (of radius r). Show what position it will rest in, and that it will be inclined to the vertical at an angle $\sin^{-1}\sqrt{\frac{4r^2-4a^2}{4r^2-3a^2}}$.

40. A sphere of radius r rests on a horizontal plane in contact with a smooth vertical rod; a heavy rod of length $2a$ attached to the vertical rod by a smooth ring rests on the sphere. Find an equation to determine θ the inclination of the rod to the vertical, and show that $\theta=45°$ if $r : a :: \sqrt{2}-1 : 2$.

41. A heavy uniform beam AB, of length $2a$ and weight W, is movable in a vertical plane round a smooth horizontal axis fixed at A. At a distance b vertically above A there is a fixed pulley C. To the end B of the beam is attached a cord, which, passing over the pulley C, sustains a body whose weight is P. If θ be the inclination of AB to AC when the system is in equilibrium, find the value of $\cos\theta$.

42. A uniform rod ACB rests against a smooth peg C, the end A being connected by a string AD with a point D, such that DC is horizontal and equal to AD. Show that if θ be the inclination of the rod to the horizon in the position of equilibrium

$$\cos\theta = \frac{a+\sqrt{a^2+32b^2}}{8b}$$

where $2a=$ length of the rod, and $b=$ length of the string.

43. In Ex. 42, show that in order that the rod may rest in the manner described $a\sqrt{3}$ must not be less than $b\sqrt{2}$.

CHAPTER XI.

MACHINES.

220. A MACHINE is an instrument by means of which pressure or motion may be transmitted from one point to another, and changed both in magnitude and direction.

All machines are either certain machines of simple construction known as the Mechanical Powers, or else combinations of one or more of these Mechanical Powers. We shall treat only of machines in equilibrium, and we shall suppose (in this Chapter) that the parts move without Friction.

221. In all machines there are two forces to be considered—(1) the *Power applied* to the machine, which we shall denote by P; and (2) the *Resistance to be overcome*, usually called the Weight, denoted by W.

In all machines the ratio $P : W$ will depend on the nature of the machine. When motion is on the point of taking place, this ratio is known as the **Modulus** of the Machine.

The ratio $W : P$ is called the **Mechanical Advantage** of the Machine.

222. The Mechanical Powers are classified as follows :—

i. **The Lever,** which includes—
  - (*a*) The Lever properly so called.
  - (*b*) The Wheel and Axle.

The point to be noticed about a Lever is that the body of the machine can move round a fixed point.

ii. **The Inclined Plane,** which includes—

(*a*) The Inclined Plane.
(*b*) The Screw.
(*c*) The Wedge.

In each of these we have a plane rigid surface.

iii. **The Pulley.**

Here we have to deal with a flexible string.

THE LEVER.

223. DEF.—A Lever is a rigid bar capable of motion about a fixed point.

This point is called the **Fulcrum**, and is denoted by *F*. The distances from the fulcrum at which the Power and the Weight act are known as the *arms* of the Lever.

The Lever is said to be a *Straight Lever* if the arms are in a straight line; if not, it is called a *Bent Lever*.

224. Levers are divided into Three Orders or Classes.

In the **First Order** *the Fulcrum is between the Power and the Weight.*

Examples.—A pair of scales; a poker when resting on a bar and used to stir the fire; a pump handle; a claw hammer when used to extract nails.

Double Levers.—A pair of scissors; a pair of pincers.

In the **Second Order** *the Weight is between the Power and the Fulcrum.*

Examples.—A crowbar when the end rests on the ground; a cork-squeezer; a wheelbarrow; an oar. (In the case of an oar we may conceive the end of the blade placed against a stone under water; the weight moved is the boat.)

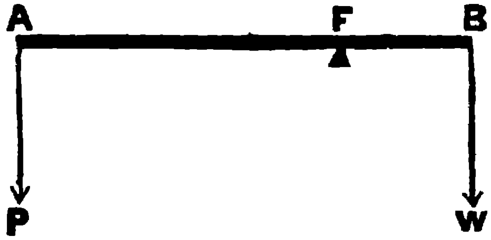
Double Lever.—A pair of nut-crackers.

In the **Third Order** *the Power is between the Weight and the Fulcrum.*

Examples.—The treadle of a lathe; the human fore-arm.
Double Levers.—A pair of sugar-tongs; a pair of spring shears.

THE CONDITION OF EQUILIBRIUM IN THE LEVER.

225. Levers of the First Order.



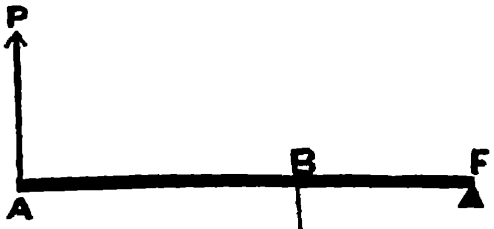
Let AB denote a straight Lever of the First Order without weight, F the position of the Fulcrum.

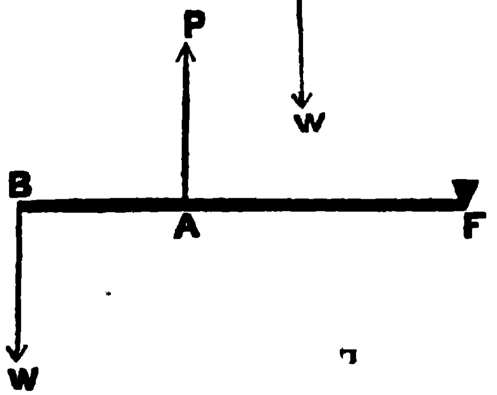
Then P and W are two like parallel forces, and these are balanced by R, the reaction of the fulcrum acting upwards. $\therefore R = P + W$. (Art. 177.)

Again, since there is equilibrium, the Algebraical Sum of Moments about F must vanish. (Art. 189.)

$\therefore$ if AB be at right angles to the lines of action of P and W, $P \times AF - W \times BF = 0$; $\therefore P \times AF = W \times BF$.

226. Levers of the Second and Third Orders.





In these cases, P and W are two unlike parallel forces.

In the Second Order—

$$P + R = W;$$
$$\therefore R = W - P.$$

In the Third Order—

$$W + R = P;$$
$$\therefore R = P - W.$$

In each case, taking Moments about F, we have, as in Art. 225,

$$P \times AF - W \times BF = 0;$$
$$\therefore P \times AF = W \times BF.$$

Therefore in all three cases—
The Moment of P about the Fulcrum = The Moment of W about the Fulcrum.

And this is known as the ‘**Principle of the Lever.**’

227. If the *Weight of the Lever* has to be considered.

Suppose its weight to be w, and let it act at a point G.

NOTE.—If the Lever be uniform it will act at its *middle* point. (See Art. 204.)

In the first Figure of Article 225, let G be situated in the arm AF.

Then $R = P + W + w$.

Taking Moments about F, we have in the First Order—

$$P \times AF + w \times GF - W \times BF = 0\,;$$

$$\therefore P \times AF + w \times GF = W \times BF.$$

In the Second and Third Orders, we have—

$$P \times AF = W \times BF + w \times GF.$$

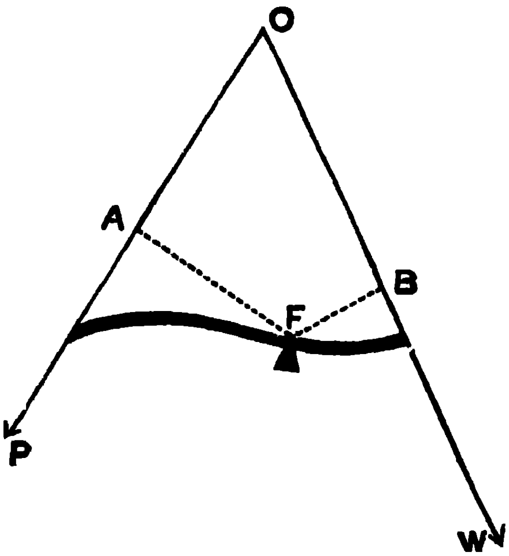
If several forces acting on a lever be in equilibrium, *the Algebraical Sum of their Moments about the Fulcrum vanishes.* (Art. 193.)

228. The Bent Lever.

If P and W be parallel, the ‘Principle of the Lever’ holds good as before.

If P and W be not parallel, let their lines of action meet in O.

Then, since there is equilibrium, the resultant of P and W must be balanced by R, the reaction of the fixed point F.



The value of R must be found by the Parallelogram of Forces.

Draw FA and FB perpendicular to the lines of action of P and W respectively.

Taking Moments about F, a point in the resultant, we have (Art. 189)—

$$P\times AF - W\times BF = 0\,; \quad \therefore P\times AF = W\times BF.$$

Thus in all cases the 'Principle of the Lever' holds good.

229. In all cases the Mechanical Advantage (Art. 221),

$$\frac{W}{P}=\frac{FA}{FB}.$$

$\therefore$ in First Order, we gain advantage by making $FA > FB$.

$\therefore$ „ Second Order we must always gain advantage.

$\therefore$ „ Third Order we must always lose advantage.

Example i.—Two forces of 10 and 15 lbs. weight act at the ends of a lever 45 inches long, and at right angles to it; find the position of the fulcrum when there is equilibrium.

Let x = distance of F from the 10 lbs.

Then $45-x$ = distance of F from the 15 lbs.;

Take moments about F—

$$\therefore 10x=15(45-x)\,; \quad \therefore 10x=675-15x\,; \quad \therefore 25x=675\,;$$

$\therefore x=27$ inches from 10.

Example ii.—Two forces act at the ends of a weightless lever, and are inclined to it at angles of 45° and 30° respectively; the arm of the first = 18 in., and of the second = 12 in. Compare the forces.

Let P and W be the forces.

Take Moments about F—

$$\therefore P\times 18\sin 45^\circ = W\times 12\sin 30^\circ\,;$$

$$\therefore \frac{P}{W}=\frac{12\times\frac{1}{2}}{18\times\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{3}.$$

Example iii.—A mass W, suspended from one end of a weightless lever 18 ft. long, is balanced by a mass of 2 lbs. at the other end; when the fulcrum is moved 9 feet it requires a mass of 20 lbs. to balance W; find the value of W.

Let x = distance of F from W in the first position;

$\therefore Wx=2(18-x)$ . . . . (1.)

Then $(x+9)$ = distance in the second position.

$\therefore W(x+9)=20(9-x)$ . . . . . . (2.)

Solving the equations (1) and (2) we obtain—

$W=10$ or 4 lbs.

NOTE.—If $W=10$, the original position of the fulcrum was 3 feet from W; if $W=4$, the fulcrum was 6 feet from W.

Example iv.—In a weightless lever of the second order, the pressure on $F=6$, and a vertical force of 18 acts at a point 8 inches from F; find the length of the lever.

$$P=W-R=12.$$

Let x = length of the lever;

Take Moments about F—

$$\therefore\ 12\times x=18\times 8\,;\quad \therefore\ x=12 \text{ inches.}$$

Example v.—A uniform straight lever weighing 2 lbs. per foot, and having bodies of 4 lbs. and 36 lbs. at its ends, balances about a point 3 feet from the greater mass; find the length of the lever.

Let x = length of the lever.

Then a weight $=2x$ lbs. acts at its middle point. (Art. 204.) Taking moments about the fulcrum, we have—

$$4(x-3)+2x\left(\frac{x}{2}-3\right)=36\times 3.$$

From which we obtain $x=12$, or -10.

Therefore, the length is 12 feet.

EXAMPLES—XXXV.

1. Enumerate the three kinds of levers, and give examples of each.

2. State the relation of P to W in each kind of lever. In which order is there mechanical advantage?

3. To what order of lever does a handspike belong?

4. To what order does a pair of sugar-tongs belong?

5. If two forces, which are not parallel, acting on a lever produce equilibrium, prove that they are inversely as the perpendiculars drawn from the fulcrum to the lines of action of the forces.

6. Investigate the relation of P to W in a uniform straight lever, taking into account the weight of the lever. What difference would it make if the lever were not uniform?

7. In a lever of the first order, 10 feet long, if $P=7$ and $W=13$, find the arms.

8. In a lever of the second order, 5 feet long, if $P=2$ and $W=10$, find the arms.

9. In a lever of the third order, 12 feet long, if $W=10$ and $P=16$, find the arms.

10. If a uniform lever of the first order be 9 ft. long and weigh 15 lbs. and if bodies weighing 45 lbs. and 20 lbs. hang at the ends, find the position of the fulcrum.

11. If the pressure on F in a weightless lever of the first order be 35 lbs. weight, and the arms are as 5 : 3, find P and W.

12. If a uniform lever of the first order, 6 feet long, weigh 12 lbs., what weight at the end of the shorter arm will balance 24 lbs. weight at the end of the other, F being 1½ ft. from the end of the lever?

13. If the pressure on F be 126 lbs. weight, and the arms of the lever (first order) are 3½ and 5½ feet respectively, find P.

14. In a weightless lever of the second order, $P=3$, $W=5$, the distance between their points of application $=4$ in., find the length of the lever.

15. A heavy uniform bar, 4 ft. long, and weighing 2 lbs., is used as a lever of the first order. Where must the fulcrum be placed in order that a weight of 1 lb. at the end of the longer arm may balance a weight of 2 lbs. at the end of the shorter?

16. Two weights, 2 and 5, balance on a lever of the first order, one of whose arms is 5 in. longer than the other; find the lengths of the arms.

17. A weight of 25 lbs. is suspended on a lever of the second order, without weight, at a point 3 ft. from F. Where must P be applied if the pressure on F be 10 lbs. weight?

18. A uniform lever of the first kind, weighing 5 lbs. per foot, rests upon F, distant 4 ft. from the end at which W acts. If P be 15 lbs. weight, W be 60 lbs. weight, find the length of the lever.

19. In a weightless lever of the second kind, 15 in. long, if the pressure on F be 6 lbs. weight, and the distance between the points of application of P and W be 10 in., find P and W.

20. A uniform straight lever weighing 2 lbs. per foot, and having 5 lbs. and 16 lbs. weight at its extremities, balances about a fulcrum 2 feet from the 16 lbs. weight; find the length of the lever.

21. The two arms of a straight lever are 36 in. and 50 in. respectively, and its weight is 10 lbs. If a weight of 58 lbs. be applied at the end of the longer arm, what weight must be applied at the end of the other to preserve the equilibrium?

22. A uniform lever weighs 50 lbs.; from its longer arm ($=10$ ft.) a body weighing 14 lbs. is suspended; what weight at the end of the shorter arm ($=6$ ft.) will keep it at rest?

23. A man carries a bundle at the end of a stick across his shoulder. At what point must the stick touch his shoulder if the pressure on it be twice the weight of the bundle?

24. Weights of 3, 6, 9, 2 lbs. respectively are placed at equal distances along a uniform rod whose weight is 6 lbs. Where must the fulcrum be placed for equilibrium?

25. The pressure on F in a lever of the first order is 30 lbs. weight; P is 5 lbs. weight, and acts at a point 5 ft. from F; find the length of the lever.

26. The pressure on F in a uniform lever of the second class, 6 feet long, and weighing 15 lbs., is 76 lbs. weight; W acts at 2 ft. from F; find the value of P.

27. A uniform heavy rod has a body weighing 5 lbs. at one end, and balances upon a fulcrum 5 ft. from that end. This weight is replaced by a weight of 10 lbs., and now the rod balances when the fulcrum is 4 ft. from that end. Find the *length* and *weight* of the rod.

28. A bar weighing 10 lbs., and 6 feet long, has its C. G. 2 ft. from one end, at which end a weight of 5 lbs. is suspended. If a weight of 9 lbs. be suspended from the other end, where must F be placed for equilibrium?

29. A bar of uniform thickness and density, whose weight is 7 lbs. and length is $5\frac{1}{2}$ feet, is used as a lever of the second kind; how far from the fulcrum must a weight of 84 lbs. be made to act, that it may be kept in equilibrium by a force equal to the weight of 14 lbs. acting upwards?

30. Weights of 10 and 15 balance on a weightless lever of the first kind; but if the fulcrum be moved 1 inch from the greater weight the equilibrium will be maintained by a weight of 12 instead of 10. Find the length of the lever.

31. A force of 100 lbs. weight acts at the end of a lever at right angles to it; at what angle must a force of 200 lbs. weight act at the same point that it may be equally effective?

32. Two forces, 6 and 8, act at the ends of a rod 12 ft. long, and are inclined to the rod at angles of 30° and 45° respectively; find position of F when there is equilibrium.

33. A weight of 20 lbs. is hung at the end A of a horizontal lever ACB, and a force of 10 lbs. weight acting at B at an angle of 30° preserves equilibrium about C. If C be distant 10 ft. from A, find the length of the lever.

34. A uniform lever AB is 6 ft. long, and weighs 15 lbs. The fulcrum is at A, and P acts at B at an angle of 45° with the lever. W is 50 lbs. weight, and acts at 4 ft. from A. Find the magnitude of P.

35. If P and Q act at the ends of a straight weightless lever (the arms of which are 5 and 7 feet long respectively), and at angles of 45° and 30° with the lever, find the ratio of P to Q.

36. Two forces, 25 and 37, act at the end of a rigid rod 12 ft. long, at angles 45° and 30° respectively; find the magnitude and position of the force which will keep the rod in equilibrium.

Bent Lever.

37. The arms of a bent lever (without weight) are at right angles, and one is three times as long as the other: a weight of 15 lbs. is applied at the end of the shorter arm, and a weight of 21 lbs. at the end of the other. Find the angle made by the latter arm with the horizon when in equilibrium.

38. The arms AC and BC of a weightless bent lever ACB, movable about a fulcrum at C, are at right angles to each other, and are 5 ft. and 7 ft. long respectively. A weight of 7 lbs. is suspended from A, and one of 10 lbs. from B; find the inclination of AC to the horizon in the position of equilibrium.

39. A uniform bent lever ACB, the arms of which are 5 ft. and 2 ft. long respectively, is suspended from C. What weight at the end of the shorter arm will cause the longer to take up a horizontal position, the weight of the lever being 70 lbs., and the angle ACB being 120°?

40. ABC is a weightless bent lever, B being the fulcrum. Weights P and Q, suspended at A and C respectively, are in equilibrium when BC is horizontal; weights P and $4Q$, similarly placed, are in equilibrium when AB is horizontal; find the angle ABC.

41. The arms of a bent lever inclined to each other at 120° are 3 ft. and 5 ft., and the weight of the lever is 24 lbs. What weight at the end of the shorter arm will cause the longer to assume a horizontal position?

42. A uniform bent lever suspended at the angle rests with the shorter arm horizontal. If the shorter arm were three times as long it would rest with the other horizontal; find the angle between the arms.

43. A uniform bent lever, when hung up at the bend, rests with the shorter arm horizontal; if the shorter arm were twice as long, it would rest with the other horizontal. Find the angle between the arms.

44. The arms of a lever are at right angles, and F is at the angle. A weight of 1 lb. is placed at the end of the longer arm, and a weight of 2 lbs. at the end of the other. If in the position of equilibrium the shorter make an angle of 30° with the vertical, find the ratio of the lengths of the arms.

45. A straight uniform rod is bent at B, so that the angle $ABC=\theta$, and $AB : BC = \sqrt{3}-1 : 1$. It is then suspended by a string from A. If BC be horizontal, show that $\theta=60°$.

Miscellaneous.

46. A weightless straight lever (l) is suspended at its middle point F, and is kept at rest by two weights which are in the ratio $m : 1$. Where must these be placed that the distance of one from F may be equal to the distance of the other from the end of the lever?

47. There are n weights W_1, W_2, W_3, . . . W_n in Geometrical Progression, and W_1 at A, one end of a lever, balances W_2 at B, the other end. Prove that a weight equal to the first $(n-1)$ weights at A will balance a weight equal to the last $(n-1)$ weights at B.

48. A man raises a cube of stone each edge of which is 3 feet long and whose weight is 2 tons, by a crowbar 4 feet long, having placed one end of the bar under the stone to a distance of 6 inches. Prove he must exert a force equal to the weight of 2½ cwt. to move the stone.

49. A man seated in a boat pulls the handle of each of a pair of sculls with force P; another man on shore holds the boat back by pulling on a horizontal rope fastened to the stern. If the distance of the rowlocks from the end of the blade of each scull be m times that of the rowlock from the hand, find the force by which the rope must be pulled.

50. Each of a pair of sculls is divided at the rowlock in the ratio 1 : 4, and a man pulls each with a force P. Another punting thrusts an oar against the bottom with a force $2P$ at an angle of 60° with the horizon. Compare their effects in propelling the boat.

51. Two weights, P and Q, balance at the extremities of a lever AB without weight; if they be interchanged, and weights x and y be added at A and B respectively, prove that, if still in equilibrium, $Py - Qx + P^2 - Q^2 = 0$.

BALANCES.

230. We shall notice only the Common Balance, the Roman or Common Steelyard, and the Danish Steelyard.

The Common Balance.

231. The Common Balance is a lever of the First kind. It consists of a beam with scale-pans suspended from its extremities. The arms of the balance are supposed to rest in a horizontal position when the pans are empty.

The following three requisites must be satisfied in a good balance:—

(i.) It must be *True.*

This condition is satisfied *if the Arms are equal.* In order that the arms should remain the same length during the oscillations of the beam, a triangular prism is fixed to the lower side, and its *knife edge* rests on a plain surface, the axis of suspension being thus reduced to a *line.* In balances constructed for very delicate weighing, the prism is of hardened steel resting on an agate surface.

It is evident that by the 'Principle of the Lever' the moments of the weights in the pans about the fulcrum are equal and opposite. Hence if correct standard 'weights' are used on a true balance, the purchaser always obtains correct measure. (See Art. 126.)

The simplest way to detect a *false* balance is to interchange the weights in the pans.

(ii.) It must be *Sensible.*

This condition is satisfied when very small differences in the weight can be detected.

It is evident that to secure sensibility, the following points must be attended to in the construction:—

1. The friction of the parts must be very small.
2. The arms must be as long as possible.
3. The beam must be as light as possible.
4. The C. G. of the instrument must be near the axis of suspension.

(iii.) It must be *Stable.*

This means that when the beam is disturbed it returns to its horizontal position.

Hence the C. G. of the instrument must be situated below the axis of suspension. (*See* Art. 200.)

It follows that *sensibility* and *stability* in the same balance are to some extent incompatible. Their relative importance must therefore depend on the purpose for which the instrument is intended. In commerce *stability* is the chief requisite, and in physical research *sensibility* is by far the more important.

232. *When the Arms are unequal the true Weight of a body is the Geometrical Mean between the false weights.*

Suppose the Arms to be a and b; let W be the *true* Weight.
Let W at the end of a appear to weigh W_1.
Let W „ b „ W_2.

$$\text{Then}\quad Wa = W_1 b$$
$$\text{and}\quad Wb = W_2 a;$$
$$\therefore\quad W^2 ab = W_1 W_2 ab;$$
$$\therefore\quad W = \sqrt{W_1 W_2}\quad \text{(Q.E.D.)}$$

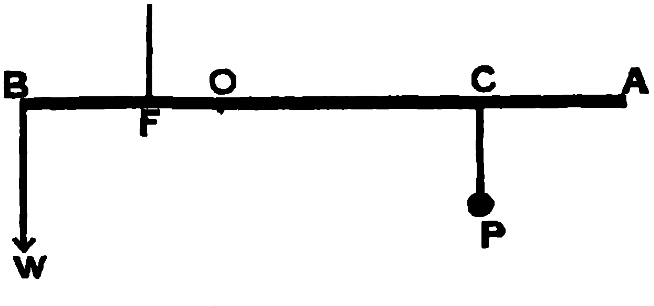
NOTE.—By Algebra we know that the Arith. Mean between two quantities is greater than the Geom. Mean.

$$\therefore\ \frac{W_1 + W_2}{2} > \sqrt{W_1 W_2}$$

Therefore the *sum of the false weights* is greater than *twice the true* weight, and hence if a trader sell from the two scales alternately he must lose.

The Common Steelyard.[1]

233. This consists of a beam AB turning round a fixed



point F. It has a scale-pan suspended at B, in which the body to be weighed is placed; and a *constant* weight P can be shifted at pleasure along the arm FA.

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[1] This is known also as the Roman Steelyard, so called from an Eastern word *Romman*, signifying a pomegranate, the form sometimes given to the mass P.

Let X be the weight of the beam and scale-pan; then X acts through G the C. G. supposed to be in the arm FB.

Let P hanging at O balance X, acting at G.

$$\text{Then } X \,.\, GF = P \,.\, OF. \quad \text{(Art. 225.)}$$

Now let a body whose weight is W be placed in the scale-pan, and let P suspended at C cause the beam to take up a horizontal position when at rest.

$$\text{Then } W \,.\, BF + X \,.\, GF = P \,.\, CF. \quad \text{(Art. 227.)}$$

$$\text{But } X \,.\, GF = P \,.\, OF;$$

$$\therefore\ W \,.\, BF + P \,.\, OF = P \,.\, CF;$$

$$\therefore\ W \,.\, BF = P(CF - OF);$$

$$\therefore\ W = \frac{OC}{BF} \,.\, P.$$

Now BF is a known distance.

If $OC = BF$, then $W = P$.
,, $OC = 2BF$, ,, $W = 2P$.
,, $OC = 3BF$, ,, $W = 3P$.

,, $OC = n \,.\, BF$, then $W = nP$.

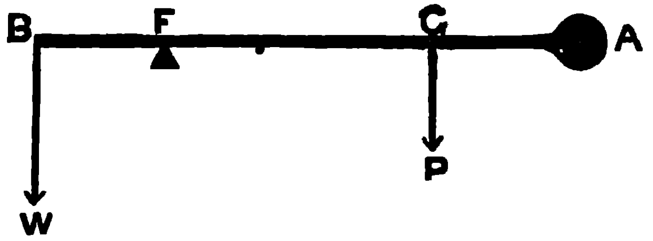
Hence, if we measure off from O towards A distances equal to BF, and mark the divisions of this scale 1, 2, 3, . . . n, we shall have *graduated* the Common Steelyard.

Example.—If P hanging at the 10th division balances W, then $W = 10P$, and if P be a known weight, W is also known.

The Danish Steelyard.

234. This consists of a straight bar terminating in a heavy knob at one end, and having a scale-pan hooked on at the other end. The peculiarity of this instrument consists in its having a *movable* fulcrum, and the Steelyard is so graduated that the weight of a body is determined by the position of F.

Let AB be the bar, and let P, the weight of the beam



and scale-pan, act at G, the C. G. When a body whose weight is W is placed in the pan the bar is pushed along until P acting at G is in equilibrium with W at B.

To graduate the Danish Steelyard.

Take moments about F.

$$\text{Then } W.BF=P.GF=P(BG-BF)$$

$$\therefore\ (W+P).BF=P.BG$$

$$\therefore\ BF=\frac{P}{W+P}.BG.$$

If $W=P$, then $BF=\frac{1}{2}BG$, and BG is a known distance.

If $W=2P$, then $BF=\frac{1}{3}BG$.

If $W=3P$, then $BF=\frac{1}{4}BG$.

. . . .

The numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}$ are in H. P.

Therefore, if the successive values of W be in A. P., the distances of F from B will be in H. P.

Example i.—The apparent weights of a body in a false balance are $4\frac{1}{2}$ lbs. and 2 lbs. respectively; find the true weight.

Let W be the true weight.

$$\text{Then } W=\sqrt{4\tfrac{1}{2}\times 2}=\sqrt{9}=3 \text{ lbs.}$$

Therefore the *true* weight = 3 lbs.

Example ii.—If a body appear to weigh p and q respectively when placed at the ends of the arms of a false balance, compare the lengths of the arms.

Let the true weight $= W$; let the length of the arms be a and b respectively.

$$\begin{aligned} &\text{Then } Wa = pb \\ &\text{and } \ \ Wb = qa \\ &\therefore \quad \frac{a}{b} = \frac{pb}{qa} \\ &\therefore \quad \frac{a^2}{b^2} = \frac{p}{q} \\ &\therefore \quad a : b = \sqrt{p} : \sqrt{q}. \end{aligned}$$

EXAMPLES—XXXVI.

1. Give a description of the Common Steelyard, and show how to graduate it.

2. A substance appears in a common balance to have weights of 9 oz. and 4 oz.; find its true weight, and compare the lengths of the arms.

3. Explain why in a well-constructed balance the C. G. must not coincide with the point of suspension.

4. A body whose true weight is 10 appears to weigh 12 in one scale of a false balance; find its apparent weight in the other scale.

5. If the arms of a false balance be in the ratio 7 : 8, what will a trader gain or lose in 112 lbs., if he sells as much from one scale as from the other?

6. If the apparent weights of a body be 4 lb. 12 oz. and 5 lbs., find the true weight.

7. State the requisites of a good balance, and how they are secured.

8. Why is it not possible to secure extreme sensibility and stability in the same balance?

9. If the real weight of a body be 12 oz., and it appears to weigh 16 oz. in one scale, find the ratio of the arms.

10. A dealer has correct 'weights,' but one arm of his balance is longer than the other. He sells an apparent weight W twice, using first one pan and then the other; what does he gain or lose?

11. Show how to graduate the Danish Steelyard.

12. In the Common Steelyard, if the graduations for 2 lbs. and 1 lb. be 3 in. apart, find the distance between the graduations for 1 lb. and ½ lb.

13. If the beam of a Common Steelyard be 14½ in. long, its C. G. 1⅜ in. from the heavier end, its weight 3 lbs., where must the fulcrum be, and what must be the distance between the graduations indicating *lbs.*, if a movable weight of 2 lbs. will just weigh up to 12 lbs.?

14. A Danish Steelyard has at one end a ball of metal 3 in. diameter and weighing 8 lbs. The bar weighs 2 lbs. and is 12 inches long; it is notched at regular intervals of ½ in. from end to end. What are the greatest and least weights which can be measured by it?

15. The arms of a balance are 6 in. and 6·1 in. respectively, the weight of the beam and scales 2 lbs.; find the real weight of the body which, when placed in the scale-pan at the end of the shorter arm, appears to weigh 3 oz., and find its apparent weight in the other scale-pan.

16. If the beam of a Common Steelyard be uniform and its weight be one-half that of the movable weight, the fulcrum being $\frac{1}{10}$ length of the beam from one end, compare the greatest weight which can be measured by the instrument with the movable weight.

17. A weight of 6 lbs. is balanced on a Common Steelyard by a movable weight of 1 lb. Both weights are then doubled. Will the equilibrium be disturbed? If so, in what direction must the movable weight be shifted to restore the equilibrium?

18. Show how the zero point is obtained on the Common Steelyard.

19. In a Common Steelyard the pounds are read off by graduations from 0 to 14, and the stones by weights hung from the extremity of the arm; if the weight corresponding to 1 stone be 7 oz., the movable weight ½ lb., and the length of the arm 1 ft., show that the distance between the graduations is ¾ inch.

20. Prove that in the Common Steelyard the distances of the lines of the scale measured from a certain point are in Arithmetical Progression, and that in the Danish Steelyard they are in Harmonical Progression.

21. In these two instruments what are the zero points respectively?

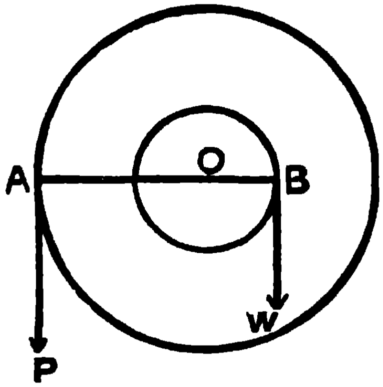
22. Determine the position in which a Common Balance will rest when loaded with unequal weights.

23. How will the stability and sensibility of the balance be affected by raising the point of suspension above the axis of the beam without lengthening the arms?

THE WHEEL AND AXLE.

235. This machine is one of the simplest and most useful modifications of the Lever. It consists of two cylinders rigidly connected, and having a common axis: the larger is called the *Wheel* and the smaller the *Axle*. At the extremities of the common axis are two pivots, which rest in fixed sockets. The Power is applied at the circumference of the Wheel, usually by means of a rope wound round its rim; and the Weight is supported by a rope coiled round the Axle in the *opposite* direction. Thus the Tensions exerted through the ropes tend to turn the machine in contrary directions.

The windlass for raising weights from the hold of a ship, a capstan for raising the anchor, a windmill, and the common water-wheel are examples where this machine is practically used.



We may suppose that the Figure represents a section of the machine made by a vertical plane at the point where the wheel and the axle are joined.

The line of action of the Weight, being vertical, will leave the axle at B. Let the line of action of P be also vertical. Then the lines of action P and W will be tangents to the circles, and therefore the line AOB will be horizontal.

We may consider AOB to be a Lever of the First Order; and, by the 'Principle of the Lever,' we have—

$$P.AO = W.BO; \text{ (Art. 225)}$$

$$\therefore \frac{P}{W} = \frac{BO}{AO} = \frac{\text{radius of the axle}}{\text{radius of the wheel}}.$$

And this is the condition for Equilibrium.

Example i.—If $P = 100$ lbs., the radius of the wheel $= 25$ inches, and the radius of the axle $= 2\frac{1}{2}$ inches, find the weight supported.

Let $W =$ weight;

$$\therefore \frac{W}{P} = \frac{\text{radius of wheel}}{\text{radius of axle}};$$

$$\therefore \frac{W}{100} = \frac{25}{2\frac{1}{2}}.$$

From which equation, $W = 1000$ lbs. weight.

Note 1.—If the thickness of the ropes is to be considered, then we must measure the radii from the common axis to the middle of the ropes.

Example ii.—If $W = 1050$ lbs., the radius of the wheel $= 3\frac{1}{4}$ feet, the radius of the axle $= 6\frac{3}{4}$ in.; find P, the ropes being each $\frac{3}{4}$ in. thick, and the power and weight being each supposed to act along the axes of the ropes.

The radius of the wheel $= (39 + \frac{3}{8})$ inches.

" " axle $= (6\frac{3}{4} + \frac{3}{8})$ "

$$\therefore \frac{P}{1050} = \frac{7\frac{1}{8}}{39\frac{3}{8}} = \frac{57}{315}.$$

From which equation, $P = 190$ lbs. weight.

Note 2.—The Pressure on the axle $= P + W$, when P and W act on opposite sides of O; and $= W - P$, when P and W act on the same side of O. (See Art. 226.)

By Article 235, $\frac{W}{P} = \frac{\text{radius of the wheel}}{\text{radius of the axle}}$.

Theoretically there is no limit to the mechanical advantage to be obtained by this machine; we have only to increase the radius of the wheel, or diminish the radius of the axle. But there are very practical limits. Thus, if the wheel be made too large, the machine becomes unwieldy; and if the axle be made too small, the machine is not strong enough to sustain the weight.

EXAMPLES.—XXXVII.

1. The radius of the axle is 6 in. and that of the wheel is 4 ft.; what weight will be sustained by a power of 24 lbs. weight, and find the pressure on the axis?

2. Find the mechanical advantage when the diameter of the wheel is 3 ft. and that of the axle is 2 in.

3. A man, pushing at the end of a pole 8 feet long, measured from centre of capstan, works a capstan whose diameter is 2 ft.; with what force must he push to overcome a resistance of 400 lbs. weight?

4. If the difference of the diameters of the wheel and the axle be 16 in., and $P : W = 1 : 2\frac{1}{3}$, find the radii.

5. A capstan is worked by a man pushing at the end of a pole. He exerts a force of 60 lbs. weight and walks 3 yards round for every 2 ft. of rope pulled in. What is the resistance overcome?

6. A man with a handle $2\frac{1}{2}$ ft. long winds a rope on an axle 16 in. diameter; the rope passes round a single movable pulley attached to the weight; find the mechanical advantage.

7. Calculate the radius of the wheel so that a power of 56 lbs. weight applied at the circumference just supports a gun weighing 12 cwt. by means of a rope, one end of which is coiled round the axle, whose diameter is 6 in., while the other end, passing round a single movable pulley, to which the gun is attached by a sling, is fixed to a hook above, the ropes on either side of the pulley being parallel.

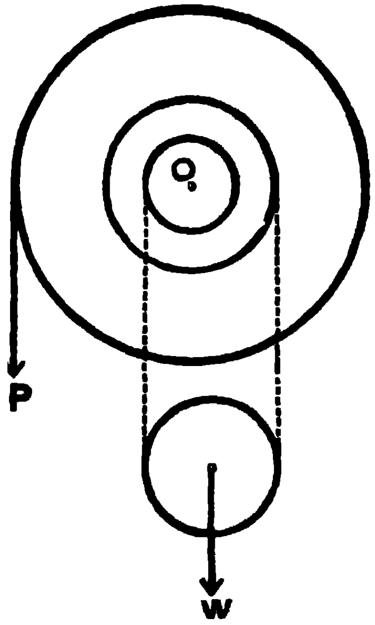
8. The radius of the wheel is 15 in., and while it makes 7 revolutions the weight of 30 lbs. is raised $5\frac{1}{2}$ ft.; find what force is required to turn the wheel. (Take $\pi = \frac{22}{7}$.)

9. If the radius of the wheel be 36 in., and radius of the axle be 3 in., the ropes being 1 inch thick, find what weight will be supported by a power of 120 lbs. weight.

236. The difficulty of obtaining a considerable mechanical advantage by the common wheel and axle is overcome by a modification of it known as the **Differential Wheel and Axle.**

The axle consists of two cylinders of different radii, these and the wheel having a common axis. The rope is coiled round the smaller cylinder, then passes under a movable pulley sustaining the weight, and finally is coiled round the larger cylinder in the contrary direction.

A section, made as before, will explain the arrangement of the ropes.



Let R = Radius of the wheel;
„ r = „ larger cylinder;
„ r_1 = „ smaller „

We may suppose the weight of the pulley to be neglected.

Let T = the Tension in rope round the pulley.

Since there is equilibrium, $2T = W$. (See Art. 239.)

An inspection of the figure will show that P and the Tension round the smaller cylinder are in equilibrium with the Tension round the larger cylinder.

Equating Moments about O, the common axis, we have—

$$P.R + Tr_1 = Tr; \quad \therefore P.R = T(r - r_1);$$

$$\text{but } T = \frac{W}{2};$$

$$\therefore P.R = \frac{W}{2}(r - r_1);$$

$$\therefore \frac{W}{P} = \frac{2R}{r - r_1}.$$

Therefore by decreasing the difference of the radii of the two cylinders, we can gain any Mechanical Advantage required.

THE PULLEY.

237. The Pulley consists of a small wheel called the ***Sheave***, which can turn round an axis, the axis being supported in a framework called the block. The wheel has a groove cut in its rim to retain in position the string by which the power is exerted.

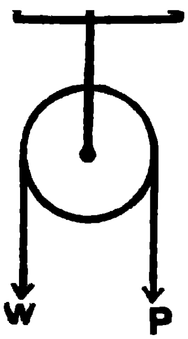
When the block can have no motion the pulley is said to be *Fixed*, otherwise it is said to be *Movable*.

Unless the contrary be stated the weights of the pulley and the strings are not taken into account, and the pulleys are assumed to move without friction.

The principle on which the equilibrium in a system of pulleys is determined may be thus stated : '*The Tension of each string employed in the system is the same throughout, and equal to the power acting at its extremity.*' (See Art. 174.)

The Fixed Pulley.

238. The tension throughout being the same, the power at one end must be equal to the weight at the other, if the system be in equilibrium.



$$\therefore P = W.$$

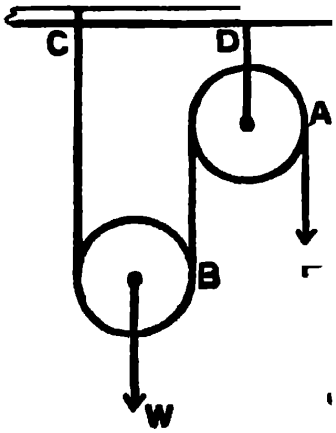
The pressure on the beam supporting the pulley $= P + W$.

The fixed pulley can therefore only change the *direction* of a force. It thus serves the same purpose as a peg.

Practically, however, it is better to use a pulley than a fixed peg, because the rotation of the wheel diminishes the friction.

The Single Movable Pulley.

239. Let a string have one end fastened to a beam C; then passing under a movable pulley B, and over a fixed pulley A, let it be pulled at the other end by a force P, which keeps in equilibrium the weight W attached to the movable pulley B.



The tension in the part of the string BC = the tension in BA, and therefore $= P$.

Case i.—*When the Strings round B are parallel.*

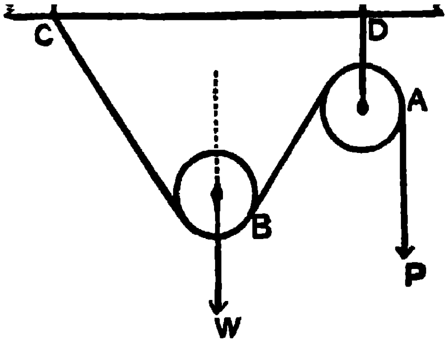
Let the Weight of the Block $=w$.

Then $(W+w)$, acting down, is balanced by two parallel forces, each of which $=P$, acting up;

$$\therefore W+w=2P.$$

The pressure on the beam in this case, and in all others, must be equal to the sum of the pressures at the points where the strings are made fast to the beam.

Case ii.—*When the Strings round B are inclined at an angle.*



Let the angle between BA and $BC=2a$.

Then the tension in each part of the string $=P$.

W is kept in equilibrium by these two equal forces; therefore the line of action of W bisects the angle $2a$ (Art. 219).

Resolving the Tensions horizontally and vertically, we have the Horizontal Components each $=P\sin a$, and these acting in opposite directions must balance; and since there is no vertical motion, we must have—

$$W+w=P\cos a+P\cos a;$$

$$\therefore W+w=2P\cos a;$$

If the weight of the pulley be neglected,

then, $W=2P\cos a$.

Example i.—Find the ratio of the power to the weight in the single movable pulley when the parts of the string are inclined to each other at an angle of 90°.

$$W=2P\cos 45°;$$

$$\therefore \frac{P}{W}=\frac{1}{2\cos 45°}=\frac{\sqrt{2}}{2}.$$

Example ii.—Find what power will sustain a weight of 150 lbs. on a single movable pulley, when the parts of the string are inclined at an angle of 120° to each other.

$$W=2P\cos a;$$
$$\therefore 150=2P\cos 60°;$$
$$\therefore 150=P;$$

Therefore the power = 150 lbs. weight.

Example iii.—If a man whose weight is 140 lbs. be suspended from a single movable pulley and support himself by holding the free end of the rope which passes over a fixed peg, with what force does he pull?

Let P=force required. His weight is supported by the tension in three strings, and the tension in each $=P$.

$$\therefore 3P=140;$$
$$\therefore P=46\tfrac{2}{3}\text{ lbs. weight.}$$

NOTE.—When a man thus pulls on a vertical rope, the downward pressure of his weight is evidently diminished by the force with which the rope pulls him up, *i.e.* by the force which he is exerting.

EXAMPLES—XXXVIII.

1. Find the ratio of P to W in the single movable pulley when the parts of the string are inclined to each other at an angle of 60°.

2. What power will sustain a weight of 800 lbs. on a single movable pulley when the strings are parallel?

3. In a single movable pulley, if $P=W$, find the angle between the strings?

4. In a single movable pulley, if $P:W=1:\sqrt{3}$, find the angle between the strings.

5. In a single movable pulley, if the weight of the pulley be 4 lbs., find what power will sustain a weight of 36 lbs., the strings being (1) parallel, (2) inclined at an angle of 30°, (3) inclined at an angle of 90°.

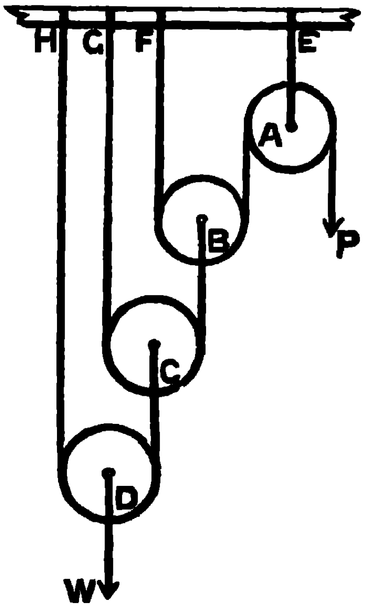
6. What power will sustain a weight of 200 lbs. on a single movable pulley when the parts of the string are at right angles to each other?

7. If $P=75$ lbs. weight, $W=119$ lbs. weight, find the angle between the parts of the string.

8. A body weighing 90 lbs. hangs at one end of a string passing over a fixed pulley, and is supported by a man weighing 11 st. pulling at the other end; find the pressure on the ground.

9. A man weighing 149 lbs. sits on a small stage weighing 10 lbs. fastened to a single movable pulley. The man has hold of the free end of the rope which passes over a peg; what force must he exert to keep equilibrium, the ropes being parallel?

240. The First System, *in which each Pulley in the system hangs by a separate string.*



The annexed figure explains the arrangement in this system. A string, at one end of which the power acts, passes over a fixed pulley *A*, under a movable pulley *B*, and is fastened to the beam at *F*. A string fastened to *B*, passes under the next movable pulley *C*, and is then made fast to the beam at *G*; and so on. The weight to be sustained is hung from the lowest movable pulley.

241. *To find the Condition of Equilibrium, when the Weights of the Pulleys are considered.*

Let w_1, w_2, w_3, be the weights of *D*, *C*, *B*, respectively.

The Tension in $DC=\frac{1}{2}(W+w_1)$, by Art. 239, Case i.

The Tension in $CB=\frac{1}{2}(\text{Tension in } DC+w_2)$

$$=\frac{W}{4}+\frac{w_1}{4}+\frac{w_2}{2}.$$

The Tension in $BA=\frac{1}{2}(\text{Tension in } CB+w_3)$

$$=\frac{W}{8}+\frac{w_1}{8}+\frac{w_2}{4}+\frac{w_3}{2}.$$

But Tension in $BA=P$;

$$\therefore P=\frac{W}{8}+\frac{w_1}{8}+\frac{w_2}{4}+\frac{w_3}{2}.$$

Next, let $w_1=w_2=w_3=w$;

$$\therefore P=\frac{W}{8}+w(\tfrac{1}{8}+\tfrac{1}{4}+\tfrac{1}{2})=\frac{W}{2^3}+w\left(1-\frac{1}{2^3}\right)$$

If there be three movable pulleys.

And, generally, if there be n pulleys of equal weight (w),

$$P=\frac{W}{2^n}+w\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots \text{ to } n \text{ terms}\right);$$

$$\therefore\ P=\frac{W}{2^n}+w\left(1-\frac{1}{2^n}\right).$$

Thus P has to keep in equilibrium both W and the Weights of the movable pulleys; in other words, the heavier the pulleys are, the *less* is the Mechanical Advantage.

If weights of pulleys be neglected, the formula becomes

$$P=\frac{W}{2^n}.$$

The Student is advised not to work the Examples by the Formula, but directly from Figures as in the following cases.

Example i.—If the number of light movable pulleys be 4, and $W=448$ lbs. weight, find P.

Denoting the tensions DC, CB, . . . by t_1, t_2, . . . we have (by Art. 239, Case i.)—

$$\begin{aligned} 2t_1&=448, && \therefore\ t_1=224;\\ 2t_2&=t_1, && \therefore\ t_2=112;\\ 2t_3&=t_2, && \therefore\ t_3=56;\\ 2t_4&=t_3, && \therefore\ t_4=28;\\ \text{But } P&=t_4; && \therefore\ P=28 \text{ lbs. weight.}\end{aligned}$$

Example ii.—If a weight of 1 ton be supported by a power of 70 lbs. weight, find the number of movable pulleys.

$$\begin{aligned} \text{Then, } 2t_1&=2240, && \therefore\ t_1=1120;\\ 2t_2&=t_1, && \therefore\ t_2=560;\\ 2t_3&=t_2, && \therefore\ t_3=280;\\ 2t_4&=t_3, && \therefore\ t_4=140;\\ 2t_5&=t_4, && \therefore\ t_5=70;\\ \text{But } P&=70; && \therefore\ t_5=P;\end{aligned}$$

$\therefore$ Number of strings $=5$; $\therefore$ Number of pulleys $=5$.

Example iii.—A weight of 96 lbs. is kept in equilibrium by a power of $9\tfrac{3}{4}$ lbs. weight when there are 4 movable pulleys of equal weight; find the weight of each pulley.

Let $x=$ Weight of each pulley;

$$\begin{aligned}
\text{Then,}\quad 2t_1&=96+x, &\therefore\ t_1&=48+\tfrac{1}{2}x\,;\\
2t_2&=t_1+x, &\therefore\ t_2&=24+\frac{3x}{4};\\
2t_3&=t_2+x, &\therefore\ t_3&=12+\tfrac{7}{8}x\,;\\
2t_4&=t_3+x, &\therefore\ t_4&=6+\tfrac{15}{16}x\,;
\end{aligned}$$

$$\text{But } t_4=P=9\tfrac{3}{4}\,;\ \therefore\ 6+\tfrac{15}{16}x=9\tfrac{3}{4}\,;$$

From which equation, $x=4$ lbs. weight.

EXAMPLES—XXXIX.

Weights of the Movable Pulleys neglected.

1. If there are 4 movable pulleys, find P, when $W=112$ lbs. weight.

2. If there are 6 movable pulleys, find P, when $W=600$ lbs. weight.

3. If there are 3 movable pulleys, find W, when $P=10$ lbs. weight.

4. If $W=640$ lbs. weight, and $P=20$ lbs. weight, find the number of movable pulleys.

5. If $P=12$ lbs. weight, and $W=768$ lbs. weight, find the number of movable pulleys.

6. Find the Mechanical Advantage when there are seven movable pulleys.

Weights of the Movable Pulleys taken into account.

7. There are 4 movable pulleys of equal weight; find the weight of each when $W=64$ lbs. weight, and $P=4$ lbs. 15 oz. weight.

8. There are 4 movable pulleys, each weighing 3 lbs.; find W, if $P=20$ lbs. weight.

9. In a system of 4 movable pulleys, show that there will be equilibrium if P, W, and the weight of each pulley be equal.

10. Will there be equilibrium on the same supposition for any number of movable pulleys?

11. There are 4 movable pulleys of equal weight; find the weight of each when $P=7\tfrac{1}{4}$ lbs. weight, and $W=88$ lbs. weight.

12. There are 4 movable pulleys, each weighing 2 lbs.; if $W=15$ lbs. weight, find P.

13. If, in Example 11, $P=10$ lbs. weight, and $W=100$ lbs. weight, find the weight of each pulley.

14. There are 6 movable pulleys, each weighing 8 oz.; find P, when $W=30$ lbs. weight.

15. If there are 3 movable pulleys, each weighing 1 lb., what weight can be supported by a power of 2 lbs. weight?

16. A power of 17·5 lbs. weight supports a weight of 1 cwt.; if there are three equal movable pulleys, find the weight of each.

17. There are 5 movable pulleys of equal weight; find the weight of each if a power of 6 lbs. weight supports a weight of 68 lbs.

18. There are three movable pulleys, the lowest weighing 1 lb., the next ½ lb., the highest ¼ lb. If $W=21$ lbs. weight, find P.

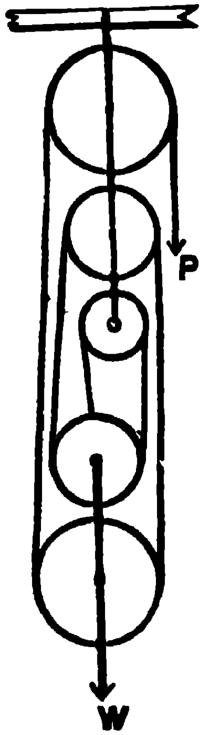
19. If the weights in the last Example be 4, 2, 1 lbs. respectively, find P, when $W=28$ lbs. weight

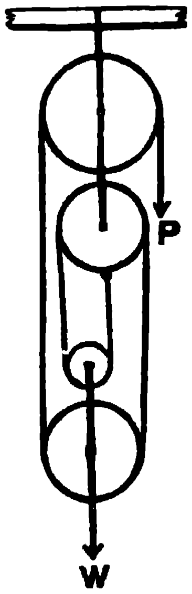
20. There are four movable pulleys; if the weights of the pulleys are in G. P., beginning at the lowest (the weight of which is w), and the common ratio is 2, find the relation between P and W.

21. There are four movable pulleys of equal weight. Find the ratio of this weight to the power, in order that the latter may just support the weights of the pulleys alone.

22. If there be four movable pulleys whose weights, beginning at the highest, are 1, 2, 4, 8 oz. respectively, find what weight will be supported by a power of 1 lb. weight.

242. The Second System, *in which the same string passes round all the pulleys.*





In this system there are two blocks, one fastened to a beam, the other movable. In the lower there are several sheaves.

If there are the same number of sheaves in each block, the string must be made fast to the fixed block, and then the number of strings (strictly parts of the one string at the lower block) is *even;* if not, it must be attached to the movable block, and then the number of strings at the lower one will be *odd.* Let there be n strings at the lower block.

To find the Condition of Equilibrium.

Since the whole weight supported is kept in equilibrium by the sum of the Tensions in these strings, and each Tension $=P$,

If the Weight of the lower Block $=w$,

then, $W+w=nP$.

Example i.—There are four sheaves at the lower block. What power will sustain a weight of ½ ton, the string being made fast to the upper block?

There will be 8 strings at the lower block;

$\therefore \quad W=8P;$

$\therefore \; 1120=8P;$

$\therefore \quad P=140$ lbs. weight.

Example ii.—If $P=3$ lbs. weight, $W=20$ lbs. weight, and there are four pulleys at the lower block, to which the string is attached, find the weight of the movable block.

There are evidently 9 strings at the lower block.

Let $w=$ Weight of the block;

$\therefore \; 20+w=9\times 3.$

Therefore, $w=7$ lbs. weight.

EXAMPLES—XI.

1. If there are ten strings at the lower block, what power will sustain 2 tons weight?

2. If there are eight strings at the lower block, find P when $W=$ 3 tons weight.

3. The string is made fast to the lower block, which weighs 20 lbs., and has four sheaves; if $W=115$ lbs. weight, find P.

4. The number of strings at the lower block is four; find the weight of the block when a power of 3 lbs. weight sustains a weight of 10 lbs.

5. There are four sheaves at the lower block; if the weight of this block be double the power, and the string is made fast to it, find the mechanical advantage.

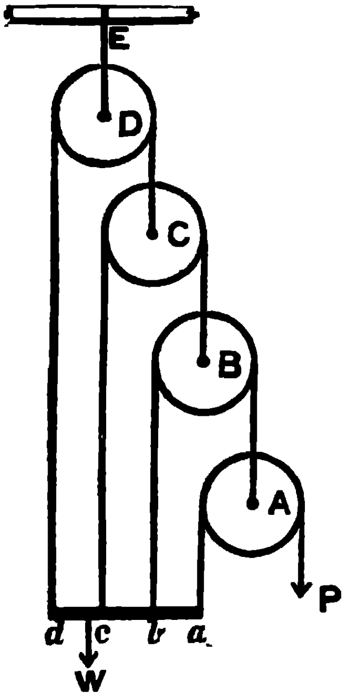
6. When $W=50$, $P=10$; and if $W=98$, $P=18$; find the weight of the lower block and the number of strings.

7. Find the number of strings at the lower block if a power of ¼ lb. weight sustain a weight of 4 lbs.

8. A man weighing 12 st. supports a weight of 6 st.; if there are seven strings at the lower block, find his pressure on the ground.

9. If there are six strings at the lower block, and the weight of the block $= 3P$, find what weight P will keep in equilibrium.

243. The Third System, *in which each string is attached to the weight.*



In this system the string passing over any pulley has one end made fast to a bar to which the weight is attached, and the other to the pulley next below it, and so on to the lowest movable pulley, one end of the string passing over which is fastened to the bar, and at the other the power is applied.

The young student will notice (i.) that the weight of the highest pulley in this system is never considered, because it is balanced by the reaction of the beam E; (ii.) that the weight of a pulley cannot affect the tension of a string passing *over* it.

244. *To find Condition of Equilibrium when the Weights of the Pulleys are considered.*

Let w_1, w_2, w_3 be the weights of A, B, C, respectively.

$$\text{Tension in } Aa = P,$$

$$\text{Tension in } Bb = 2 \times \text{Tension in } Aa + w_1 = 2P + w_1,$$

$$\text{,, } \quad Cc = 2 \times \text{Tension in } Bb + w_2 = 4P + 2w_1 + w_2,$$

$$\text{,, } \quad Dd = 2 \times \text{Tension in } Cc + w_3 = 8P + 4w_1 + 2w_2 + w_3;$$

$$\text{But } W = \text{Sum of the Tensions.}$$

$$\therefore W = 15P + 7w_1 + 3w_2 + w_3$$

$$= (2^4 - 1)P + (2^3 - 1)w_1 + (2^2 - 1)w_2 + (2^1 - 1)w_3.$$

$$\text{Let } w_1 = w_2 = w_3 = w.$$

$$\text{Then, } W = (2^4 - 1)P + (2^3 + 2^2 + 2^1 - 3)w,$$

if there are four pulleys.

Similarly, $W=(2^5-1)P+(2^4+2^3+2^2+2-4)w$,
if there are five pulleys.

And, generally, if there be n pulleys of equal weight (w),

$$W=(2^n-1)P+(2^{n-1}+2^{n-2}+\ldots\overline{n-1}\text{ terms}-\overline{n-1})w,$$

$$\text{or, } W=(2^n-1)P+(2^n-n-1)w.$$

In the Third System the weights of the movable pulleys assist the Power, and thus the heavier these are, the *greater* is the Mechanical Advantage.

If the weights of the pulleys be neglected, the formula becomes $W=(2^n-1)P$.

Example i.—What power will sustain a weight of 84 lbs. by three pulleys, each cord being attached to the weight, the weights of the pulleys being neglected?

Tension in $Aa=P$,
,, $Bb=2P$,
,, $Cc=4P$;
$\therefore$ Sum of Tensions $=7P$;
But ,, ,, $=84$; $\therefore$ $7P=84$;
$\therefore$ $P=12$ lbs. weight.

Example ii.—If $W=313$ lbs. weight, and there are three *movable* pulleys, each weighing 8 lbs., find the magnitude of P.

Tension in $Aa=P$, (Fig. Art 243.)
,, $Bb=2P+8$,
,, $Cc=4P+16+8=4P+24$,
,, $Dd=8P+48+8=8P+56$;
$\therefore$ $313=15P+88$; $\therefore$ $P=15$ lbs. weight.

Example iii.—What weight can be supported by a power $=25$ lbs. weight, if there be four movable pulleys, each weighing 5 lbs.?

Tension in $Aa=25$ lbs. weight.
,, $Bb=50+5=55$ lbs. weight.
,, $Cc=110+5=115$,,
,, $Dd=230+5=235$,,
,, $Ee=470+5=475$,,
$\therefore$ $W=25+55+115+235+475$; $\therefore$ $W=905$ lbs. weight.

If we use the formula of Article 244,

$W=(2^5-1)\times 25+(2^5-5-1)\times 5=775+130=905$ lbs. weight, as before.

Example iv.—There are three pulleys, two of them movable, each weighing 1 lb., and 2 inches in diameter. The last string is made fast at both ends to the bar, which supports a weight of 12 lbs. At what distance from the first string must the weight be made fast to the bar?

In the figure of Article 243, consider b and a to be the ends of the last string.

It is evident that the bar ad is 4 inches long.

Let the Tension in $Ba = T$;
Then ,, $Bb = T$,
,, ,, $Cc = 2T + 1$,
,, ,, $Dd = 4T + 2 + 1$;

$$\therefore\ 12 = 8T + 4;$$
$$\therefore\ T = 1.$$

Then 12 is sustained by the parallel forces 7, 3, 1, 1 acting up, and in order that the bar may remain in a horizontal position, the 12 lbs. must be fastened to the point where the line of action of the resultant cuts the bar.

To find this point take moments about d, and let x = distance from d at which the resultant acts.

$$\therefore\ 12x = 7 \times 0 + 3 \times 1 + 1 \times 2 + 1 \times 4;$$
$$\therefore\ 12x = 9.$$

Therefore $x = \frac{3}{4}$ inch from the first string, and at this point therefore the 12 lbs. must be suspended.

EXAMPLES—XLI.

Weights of the Movable Pulleys neglected.

1. If there are five pulleys, find P, when $W = 93$.

2. If $W = 315$, and $P = 5$, find the number of pulleys employed in the system.

3. When $W = 140$, and the number of pulleys is three, find the magnitude of the power.

4. When $W = 300$ and the number of pulleys is four, find P.

5. A weight of 12 cwt. is supported by four ropes; find the power applied at the free end of the last rope.

6. If there are three pulleys, and, in the position of equilibrium, the string round the middle one is nailed to it, show that the tension of the string going from this to the weight is $\frac{1}{2}(W - 3P)$.

Weights of the Movable Pulleys taken into account.

7. If $W = 86$ lbs., find the magnitude of P, when there are three movable pulleys, each weighing 1 lb.

8. If there are three movable pulleys, the lowest weighing 4 lbs., the next 2 lbs., the third 1 lb., find P when $W = 80$ lbs. weight.

9. There are five pulleys of equal weight, and $W = 44P$; compare the power with the weight of a pulley.

10. If there are four movable pulleys, each weighing 2 lbs., find P when $W = 7$ cwt. 2 qrs. 22 lbs.

11. If P = weight of each movable pulley, find W, when there are five pulleys in the system, each weighing 1 oz.

12. There are four movable pulleys, each weighing 1 lb.; find W, if $P = 26$ lbs. weight.

13. Three movable pulleys, each weighing 12 lbs., are used in this system to sustain a ton weight; find the power.

14. If the power be equal to the weight of the lowest pulley, and if each pulley weighs three times as much as the one immediately below it, prove that the weight of each pulley is equal to the tension of the string passing over it.

15. If there be five light pulleys, each 6·2 inches in diameter, the highest being fixed, and the weight supported be a uniform bar, show that the string which passes round the highest movable pulley must be attached to the bar at a distance of $\frac{1}{2}$ inch from its middle point.

MISCELLANEOUS EXAMPLES.—XLII.

Pulleys.

1. A rope fixed at one end passes under a movable and over a fixed pulley, and the free end hangs down. A man whose weight is 12 st. sits in a sling made fast to the movable pulley, and supports himself by holding on to the free end. What force does he exert if the strings are all vertical?

2. To the lower block in the second system, at which 8 strings act, a platform is fastened, and a man whose weight is W stands upon it. If the weight of the block and platform be $\frac{1}{8}W$, find what fraction of that weight he must exert upon the free end of the string in order to support himself.

3. In the Second System there are four sheaves in each block. Find (1) the greatest weight which a man weighing 12 st. can raise; (2) his pressure on the ground when he supports a weight of 6 st.

4. A movable block of n sheaves supports a basket in which a man sits whose weight is W. Find (1) the weight which must be fastened to the end of the rope to support him; (2) the strength with which he must pull to support himself, the weight of the basket being neglected.

5. In the Second System a platform, which together with the block weighs 300 lbs., is suspended from the lower block, and a man whose weight is 140 lbs. standing on the platform, exerts sufficient force to maintain the equilibrium; find that force, if there be 10 strings at the lower block.

6. A man weighing 9 st. stands on a platform attached to the lower block of the Second System, and supports himself by pulling the free end of the string which hangs down, the strings being parallel; what force must he exert?

7. A power P supports a weight W by a single movable pulley, the strings being inclined at an angle 2α; if the strings were inclined at an angle α, P would support 6 lbs. weight. Find α.

8. In that system of pulleys where each hangs by a separate string, show that if P be the power, W the weight, w the weight of each pulley, and n the number of movable pulleys, $P-w=\frac{1}{2^n}(W-w)$.

9. A string fastened at A passes under a movable pulley bearing a weight P; it then passes over a fixed pulley at B, and under a second movable pulley bearing a weight Q, and is fastened to a peg at C. Find the tension of the string, when the angle at one of the movable pulleys is *double* that at the other. ($P>Q$.)

10. Draw the figures of the First and Third Systems.

11. If there are three pulleys (each hanging by a separate string), and all the strings are parallel except that passing round the highest pulley, the parts of which include an angle of 120°, show that $4P=W$.

12. A weight is suspended by three movable pulleys arranged according to the First System, and another weight by three pulleys arranged according to the Third System. The two weights are joined, and the free ends of the strings are tied, all the strings being vertical. The sum of the two weights $=W$. Find the tension of the middle string, the weights of the pulleys being neglected.

13. W is the weight of the lower block in that system where the same string passes round all the sheaves, n is the number of strings at this block. Show that there is no mechanical advantage when the weight is equal to, or less than, $\frac{1}{n-1}W$.

14. In that system where the same string passes round all the pulleys, show that for the different parts of the string to be parallel the radii of the pulleys must be in A. P.

15. A ton of iron is kept in equilibrium by a weight made fast to a string passing over a fixed and under a movable pulley. If the weight of each be 1 lb., find the whole weight sustained by the beam to which the strings and fixed pulley are attached.

16. Three pulleys, each $3\frac{1}{2}$ inches in diameter, are arranged according to the Third System, the highest being fixed. Strings passing round them are made fast to, and kept in equilibrium by, a horizontal bar of metal. At what distances from its middle points are the strings severally fastened?

17. If w_2, w_3 be the two lower pulleys in the Third System, in which w_1 is the fixed pulley, and these are of such weights that, supposing them to be interchanged, P would be diminished by one-half, prove that

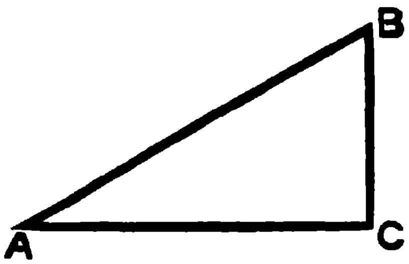
$$w_2=\frac{4W-7P}{16}, \text{ and } w_3=\frac{4W-35P}{16},$$

W being the weight supported.

18. A weight of 360 lbs. has to be supported by 4 movable pulleys arranged according to the First System. The weights of the pulleys are 2, 3, 4, 6 lbs. respectively. In what order must they be placed that P may be the *least* possible?—and find that value of P.

THE INCLINED PLANE.

245. **An Inclined Plane,** as a Mechanical Power, means a rigid plane which makes an angle less than a right angle with the horizon, and this angle is known as the *inclination of the plane.*



Let AC be any horizontal line, AB any line inclined to AC; if from any point B in AB a perpendicular BC be drawn to AC, then if these lines lie in planes at right angles to the page of the book, and the page be so held that the plane containing AC be *horizontal*, the plane containing AB will be the Inclined Plane intended. Taking the right-angled triangle ABC, a vertical section through these planes, AB, is called the *length*, AC the *base*, and BC the *height* of the Inclined Plane.

We assume that the plane is perfectly *rigid*, and therefore capable of supporting any pressure exerted at right angles to its surface, that the plane is perfectly *smooth*, and that the lines of action of all the forces lie in the same vertical plane.

When a body, whose weight $= W$, rests in equilibrium on a smooth inclined plane, there are three forces in action—

(1) The Weight $(= W)$ acting vertically downwards.

(2) The Reaction $(= R)$ of the plane acting at right angles to its surface.

(3) The Power $(= P)$ acting in some assigned direction.

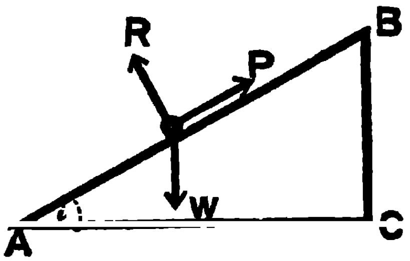
246. It is usual to discuss *three* cases.

Case i.—When the direction of P is parallel to the plane.

Case ii.—When the direction of P is horizontal.

Case iii.—When the direction of P makes any angle with the plane.

247. Case i.—*To find the Condition of Equilibrium on the Inclined Plane when the Power acts along the Plane.*



Let the inclination $BAC = i$. Let a body, whose Weight $= W$, be supported by a Power $= P$, acting parallel to the plane.

Let $R =$ the Pressure of the body on the plane, and therefore $=$ the Pressure of the Plane on the body, by Third Law of Motion. *Resolve the forces along the plane*—

$$\text{Then, } P = W\cos(90^\circ - i);$$

$$\therefore P = W\sin i \quad \ldots\ldots \quad (1.)$$

Resolve the forces at right angles to the plane—

$$\text{Then, } R = W\cos i \quad \ldots\ldots \quad (2.)$$

Equation (1) gives the Power necessary to maintain the Equilibrium.

Equation (2) gives the Pressure on the Plane.

These relations can also be obtained by Lami's Theorem.

Example i.—A plane rises 3 in 7;* what force along the plane will sustain a weight of 2 tons?

$$\text{We have } P = W\sin i.$$

$$\therefore P = 2 \times 2240 \times \frac{3}{7};$$

$$\therefore P = 1920 \text{ lbs. weight.}$$

Example ii.—If the power act along the plane, and the Mechanical Advantage $= 2$, find W, when $R = \sqrt{3}$.

$$\text{From Equation (1),} \quad \frac{W}{P} = \operatorname{cosec} i;$$

$$\therefore \operatorname{cosec} i = 2; \quad \therefore i = 30^\circ.$$

$$\text{From Equation (2),} \quad R = W\cos 30^\circ;$$

$$\therefore \sqrt{3} = W.\frac{\sqrt{3}}{2}; \quad \therefore W = 2.$$

---

* A gradient is sometimes measured in terms of the *tangent* of the inclination. In this book, it is measured in terms of the *sine*.

EXAMPLES—XLIII.

1. A weight of 40 lbs. is supported on an inclined plane, the angle of inclination being 30°; find (1) the power acting along the plane, and (2) the pressure on the plane.

2. A body weighing 50 lbs. is supported by a power of 25 lbs. weight acting along the plane; find (1) the angle of inclination, (2) the pressure on the plane.

3. A weight resting on a smooth inclined plane, the angle of inclination being 30°, is kept at rest by a weight hanging freely; compare the two weights.

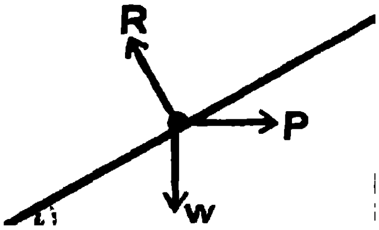
4. Find the angle of inclination, when a force of 15 lbs. weight, along the plane, supports a weight of 30 lbs.

5. If $W=40$, and $\tan i = \cdot 75$, find the force along the plane which will keep W in equilibrium.

6. If, when P acts along the plane, $P=R$, find (1) the angle of inclination; (2) the Mechanical Advantage.

7. A body whose weight is 400 lbs. is kept at rest by means of a rope on a plane which rises 1 in 40; find the tension of the rope.

248. Case ii.—*To find the Condition of Equilibrium on the Inclined Plane when the Power acts horizontally.*



Resolve the forces along the plane—

Then $P\cos i = W\cos(90° - i)$;

$$\therefore P = W\tan i \quad . \; . \; . \quad (1.)$$

Resolve the forces at right angles to the plane—

Then $R = W\cos i + P\sin i$.

Substitute for P its value $W\tan i$;

$$\therefore R = W\cos i + W\tan i \,.\, \sin i$$

$$= W\frac{(\cos^2 i + \sin^2 i)}{\cos i};$$

$$\therefore R = W\sec i \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

Equation (1) gives the Power which will keep the body in equilibrium; Equation (2) gives the Pressure on the plane.

These equations can also be obtained by Lami's Theorem.

Example i.—If a force of 15 lbs. weight, acting horizontally, keep a body at rest on a plane whose inclination is 60°, find the weight of the body.

By Equation (1), $P = W \tan i$;

$\therefore\ 15 = W.\ \sqrt{3}$.

Therefore, $W = 5\sqrt{3}$ lbs. weight.

Example ii.—If $R = 2$ lbs. weight, P acting horizontally $= 1$ lb. weight, find both the value of i and of W.

From (1) and (2) $\frac{P}{R} = \sin i$;

$\therefore\ \sin i = \frac{1}{2}$;

$\therefore\ i = 30°$.

Then, by Equation (1), $P = W \tan 30°$;

$\therefore\ 1 = W.\frac{1}{\sqrt{3}}$.

Therefore, $W = \sqrt{3}$ lbs. weight.

EXAMPLES—XLIV.

1. If W be 12, and P act horizontally, find the magnitude of P when R is 24 lbs. weight.

2. What force acting horizontally is necessary to support a weight of 100 lbs. on a plane inclined at an angle of 30° to the horizon?

3. What force acting horizontally will keep in equilibrium a weight of 15 lbs. on an inclined plane, and produce a pressure of 17 lbs. weight on the plane?

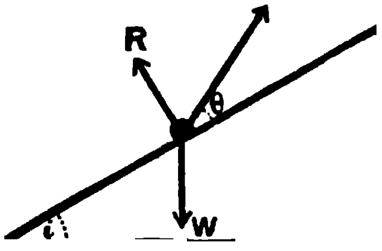
4. If W be 50 lbs. weight, and the base and length of the plane be respectively 8 ft. and 10 ft., find the value of P acting horizontally.

5. If W be 60 lbs. weight, and the height and base of the plane be respectively $3\frac{1}{2}$ ft. and 12 ft., find the magnitude of both R and P, the latter acting horizontally.

6. A weight of 286 is supported on an inclined plane by a horizontal force of 48; find the inclination of the plane.

7. If R be 50, and P, acting horizontally, be 20, find W and the inclination of the plane.

249. Case iii.—*To find the Condition of Equilibrium on the Inclined Plane when the Power is inclined at any angle to the Plane.*



Resolve the forces along the plane—

Then, $P\cos\theta = W\cos(90^\circ - i)$;

$$\therefore P = \frac{W\sin i}{\cos\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1.)$$

Resolve the forces at right angles to the plane—

Then, $R + P\sin\theta = W\cos i$.

For P substitute in this equation its value from (1);

$$\therefore R + \frac{W\sin i}{\cos\theta} . \sin\theta = W\cos i;$$

$$\therefore R = \frac{W(\cos\theta\cos i - \sin\theta\sin i)}{\cos\theta};$$

$$\therefore R = \frac{W . \cos(\theta + i)}{\cos\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

Equation (1) gives the Power which, acting at an angle θ with the plane, will keep the body at rest.

Equation (2) gives the Pressure on the plane.

These Equations, particularly (2), can be very easily obtained by Lami's Theorem.

Example.—What force acting at any angle of 30°, with a plane whose inclination is 45°, will keep a body weighing 150 lbs. at rest on the plane?

Let P = the force required.

By Equation (1), $P\cos 30^\circ = 150\sin 45^\circ$;

$$\therefore P = \frac{150 \times 2}{\sqrt{3} \times \sqrt{2}} = \frac{300}{\sqrt{6}} = 50\sqrt{6} \text{ lbs. weight.}$$

Note.—The following Examples require the aid of Logarithmic Tables for their solution.

EXAMPLES—XLV.

1. What power acting at an angle of 22°, with a plane whose inclination is 38°, will keep a weight of 150 lbs. at rest on the plane?

2. A weight of 45 lbs. is supported on an inclined plane by a power of 18 lbs. weight; if the line of action of the power make an angle of 12° with the plane, find the inclination of the plane.

3. The inclination of an inclined plane $=27^\circ\ 15'$, and P, inclined to the plane at an angle $=14^\circ\ 16'$, keeps a weight of 67 lbs. at rest upon it; find P.

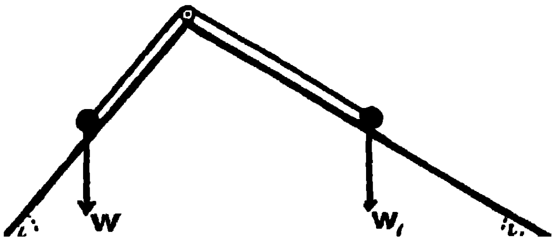
4. If $W=286$, $P=136$, $\theta=23^\circ\ 28'$, find the inclination of the plane.

5. Find the greatest weight which can be supported on a plane whose inclination is 42°, if the power of 74 lbs. weight act at an angle of 60° with the horizon.

6. A body weighing 3 cwt. is supported on a plane, inclined at an angle of 30° to the horizon, by a force of 230 lbs. weight whose line of action makes an angle θ with the plane; find θ.

THE DOUBLE INCLINED PLANE.

250. Let two bodies whose weight are W and W_1 rest on two planes whose angles of inclination are i and i_1 respectively, and be connected by a string passing over the common vertex.



Let $T=$ the Tension in each part of the string. (Art. 174.)

Consider the equilibrium of W;

Then, $T=W\sin i$ (Case i.)

Consider the equilibrium of W_1;

Then, $T=W_1\sin i_1$;

Therefore $W\sin i=W_1\sin i_1$; or, $\dfrac{W}{W_1}=\dfrac{\sin i_1}{\sin i}$.

That is, the *Weights are inversely as the sines of the angles of inclinations, and therefore directly as the lengths of the planes.*

Example.—Two weights W and 25 rest on planes whose inclinations are 45° and 60° respectively, and are connected by a string passing over the common vertex; find W.

$$W \sin 45^\circ = 25 \sin 60^\circ;$$

$$\therefore\ W = \frac{25\sqrt{3}}{2} \times \frac{\sqrt{2}}{1} = \frac{25\sqrt{6}}{2}.$$

Therefore $W = 30{\cdot}5$ lbs. nearly.

EXAMPLES XLVI.

1. If the weights W and W_1 be moved along the planes, the string keeping taut, prove that the vertical distances through which they move will be inversely as the weights.

2. $P = 75$ rests on a plane whose inclination $= 53^\circ\ 20'$, and is balanced by Q resting on a plane whose inclination $= 29^\circ\ 50'$; find Q.

3. Weights of 20 lbs. and 40 lbs. are in equilibrium on a double inclined plane; the angle of less inclination is 25°; find the other inclination, and also the tension in the string.

4. Two inclined planes meet at a right angle; if $i = 30^\circ$, and $i_1 = 60^\circ$, and two weights, connected by a string over the common vertex, are in equilibrium, show that the pressure on one plane $= 3$ times the pressure on the other plane.

5. Two inclined planes are of the same height, and are 8 feet and 5 feet long respectively. A weight of 20 oz. rests on the shorter plane; find the weight which, when placed on the longer plane and connected with the other weight by a string over the common vertex, will keep it at rest.

6. If two weights support each other on a double inclined plane by means of a string passing over the common vertex of the planes, show that if, the string being taut, the bodies are moved on the plane, the C. G. of the weights moves in a horizontal line.

MISCELLANEOUS EXAMPLES—XLVII.

Inclined Plane.

1. A body is kept in equilibrium on a smooth inclined plane, (1) by a force acting horizontally; (2) by a force acting in a direction parallel to the plane. Compare these two forces.

2. Find the Mechanical Advantage when P makes an angle θ with the plane.

3. If a force of 30 lbs. weight acting horizontally will keep a body at rest on a plane inclined at an angle of 30° to the horizon, find what force acting along the plane will keep the body at rest.

4. If R_1 be the pressure on the plane when a body whose weight is W is supported by a force acting along the plane, and R_2 be the pressure when the force acts horizontally, show that $W=\sqrt{R_1R_2}$.

5. A weight is supported on a plane inclined to the horizon at an angle α, by a force P, which makes an angle β with the plane. If $R=P$, show that $\beta=90-2\alpha$.

6. If a weight W resting on a plane, whose inclination is i, be acted on by a horizontal force, $W\sin i$, what additional force along the plane is necessary to keep the body from moving?

7. It takes three times the power to support a given weight on an inclined plane ABC when placed on the side AC that it does when on the side BC; find the least power by which a weight of 100 lbs. may be supported on the plane.

8. A weight is supported on a smooth inclined plane by a horizontal force equal to the weight. Compare the pressure with that which would be exerted if the weight were supported by the least possible force.

9. A body whose weight is W is supported by P acting horizontally. If the inclination were doubled, $\frac{3}{5}W$ would be supported by P along the plane; find the inclination.

10. If P be the force which acting along the plane can support a given weight, and P_1 be the force which when supporting it is equal to the pressure on the plane, show that P could support a weight P_1 on a plane of twice the inclination.

11. A weight can be supported by P acting along a plane, or by $2P$ acting horizontally. What force, acting at an angle of 45° with the plane, would support it? If R_1 and R_2 be the pressures in the first two cases, prove that $R_1R_2=\frac{4}{3}P^2$.

12. On a smooth inclined plane W is supported by P and the pressure is R. If $R:P:W$ as $4:5:7$, find (1) the angle of inclination, and (2) the angle which P makes with the plane.

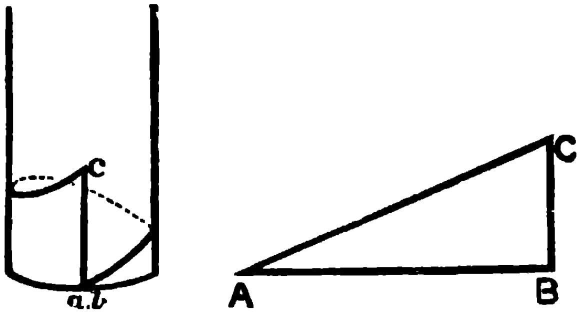
13. Apply the proposition known as the Triangle of Forces to obtain the condition of equilibrium on a smooth inclined plane.

14. P and Q, acting respectively along the plane and horizontally, will each singly support a weight W on a smooth plane; show that $PQ=W\sqrt{Q^2-P^2}$.

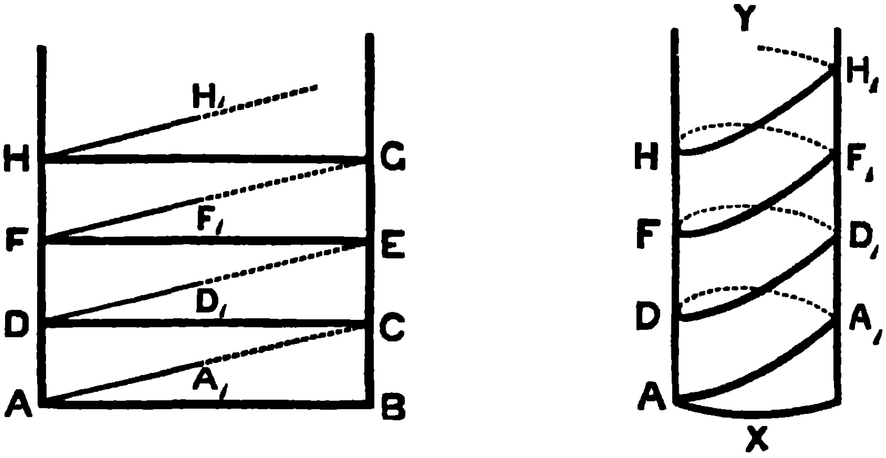
15. A force P, acting along the plane, can support a weight W, and acting horizontally can support a weight W_1; show that $P^2=W^2-W_1^2$.

THE SCREW.

251. If a right-angled triangle *ABC*, whose base *AB* is equal to the circumference of a cylinder, be wrapped round the cylinder, with the base *AB* horizontal, then the points *A* and *B* will coincide at *a*, *b*, and the hypothenuse will take the form of a curve; *cb* will then be the height *CB* of the triangle.



If we take a rectangle *ABGH*, and divide its sides *AH*, *BG* into equal parts, and if lines be drawn as in the figure, we divide the surface into triangles similar in all respects to the right-angled triangle *ABC*. If this rectangle be now wrapped round a cylinder *XY*, whose circumference is equal to *AB*, then it is evident that the hypothenuses *AC*, *DE*..... will form a continuous line, *C* coinciding with *D*, *E* with *F*, and so on. The broken parts of *AC*, *DE*..... will be at the back of the cylinder as represented in the figure. The spiral so traced on the cylinder is the form of the screw when the *thread* of the latter is reduced to a line.



This thread makes a constant angle with the horizon, or the base of the cylinder: we shall denote it by i. The height of the triangle, or inclined plane ABC, thus wrapped round the cylinder, is equal to the distance between the threads *measured parallel to the Axis of the Cylinder.*

$$\text{It is evident that } \frac{CB}{AB} = \tan i;$$

$$\textit{i.e.}\ \frac{\text{distance between the threads}}{\text{circumference of the cylinder}} = \tan i.$$

252. If we suppose the body of the cylinder cut away to a certain depth, leaving a protuberance of equal thickness along the spiral, we have a screw. The under surface of a circular spiral staircase will give the student a clear idea of the appearance that would be presented by a square-threaded screw. The cylinder fits into a fixed block in which a groove is cut, and in which the thread of the screw works. This groove is known as the *companion screw.* The only motion which the cylinder can have in the direction of its axis must be the result of causing the cylinder to rotate round its axis. This is usually effected by a force acting at the end of a lever (called the *Power-arm*) at right angles to the axis.

253. *Condition of Equilibrium on the Screw.*

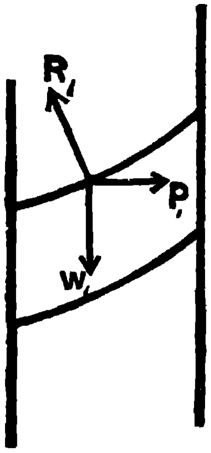
We assume that the axis is vertical.

The forces in equilibrium are—

(1) The Power acting horizontally at the end of a lever $\equiv P$;

(2) The Sum of the Reactions along the companion screw at right angles to the thread $\equiv R_1 + R_2 + \ldots$

(3) The Reaction caused by the cylinder pressing against some object, in the direction of its axis $\equiv W$



We have thus **Case ii.** of the Inclined Plane—

Resolve the forces Vertically and Horizontally.

The *vertical* Components are $R_1 \cos i$, $R_2 \cos i$, . . and these are balanced by W.

The *horizontal* Components are $R_1 \sin i$, $R_2 \sin i$, ... and the sum of the moments of these about the axis in one direction is balanced by the moment of P in the opposite direction.

Equating these moments, we have (if length of Power-arm $= a$ and the radius of the cylinder $= r$)—

$$Pa = (R_1 + R_2 + \ldots\ldots) \, r \sin i \quad \ldots\ldots \quad (1.)$$

$$\text{and } W = (R_1 + R_2 + \ldots\ldots) \cos i \quad \ldots\ldots \quad (2.)$$

Dividing (1) by (2) we have—

$$\frac{Pa}{W} = r \tan i; \quad \therefore \frac{P}{W} = \frac{r \tan i}{a} = \frac{2\pi r \tan i}{2\pi a}.$$

Now, $2\pi r$ = circumference of cylinder,

$2\pi a$ = circumference of circle described by P,

$$\tan i = \frac{\text{distance between the threads}}{\text{circumference of cylinder}};$$

$$\therefore \frac{P}{W} = \frac{\text{circumference of cylinder}}{\text{cir. of circle described by } P} \times \frac{\text{dist. between threads}}{\text{cir. of cylinder}};$$

$$\therefore \frac{P}{W} = \frac{\text{distance between the threads}}{\text{circumference of circle described by } P}.$$

And this is the condition of Equilibrium.

Theoretically, there is no limit to the Mechanical Advantage which may be obtained by this machine; we have only to diminish the interval between the threads, or increase the length of the power-arm. Practically, however, if we make the threads too thin they cannot endure a great pressure, and if we make the arm too long the machine becomes unwieldy.

Example i.—Find the Mechanical Advantage on a screw, the threads of which are $\frac{2}{3}$ in. apart, the length of the power-arm being 12 inches.

$$\frac{W}{P} = \frac{\text{circumference of circle described by } P}{\text{distance between the threads}};$$

$$\therefore \frac{W}{P} = \frac{2 \times \frac{22}{7} \times 12}{\frac{2}{3}}.$$

From which equation, $\frac{W}{P} = 113$ nearly.

Example ii.—If the thread make 33 turns in 4 inches, what power at the end of a lever 21 inches long will sustain a weight of 1000 lbs.?

$$\frac{P}{W}=\frac{\text{distance between threads}}{\text{circumference of circle by } P};$$

$$\therefore \frac{P}{1000}=\frac{\frac{4}{33}}{2\times\frac{22}{7}\times 21}.$$

From which $P=\cdot 91$ lb. weight.

Examples—XLVIII.

1. The diameter of a screw is 7 in., and the distance between the threads is $\frac{1}{3}$ in.; find the Mechanical Advantage.

The force is applied at the rim of the bolt.

2. The diameter of a screw is 4 in., and the distance between the threads is $\frac{2}{5}$ in.; find the Mechanical Advantage.

3. Find the weight which can be sustained by a power of 1 lb. weight acting at a distance of 4 ft. 8 in. from the axis of the screw, the distance between the threads being 1 inch.

4. If, in the last example, the distance between the threads be $2\frac{1}{2}$ inches, find the weight supported.

5. If 5 turns of a screw cause it to advance $\frac{3}{4}$ inch in the direction of its length, what power at the end of a lever 2 feet long will cause the machine to exert a pressure of $\frac{1}{2}$ ton weight?

6. Find the length of the lever at the extremity of which a force of 1 lb. weight will support a weight of 80 lbs. on a screw, the distance between the threads being $\frac{3}{4}$ in.

7. Find the inclination to the horizon of the thread of a screw, which, with a force of 5 lbs. weight acting at the end of a lever 2 ft. long, can support a weight of 60 lbs., the radius of the screw being 2 inches.

8. The distance between the threads being $\frac{1}{2}$ inch, find W, if P be a force of 3 lbs. weight acting at $1\frac{1}{2}$ yards from the axis of the screw.

9. Find length of the power-arm, if $P=2$, $W=1000$, and the distance between the threads $=\frac{3}{4}$ in.

10. The angle of a screw is 15°, the radius of the cylinder is 3 in., the length of the power-arm is $2\frac{1}{2}$ ft.; find P, when $W=150$ lbs. weight.

11. If W be 112 lbs. weight, the length of the power-arm 1 ft., and the screw advance 12 inches in 96 turns, find P.

12. The angle of a screw is 30°, the length of the lever is five times the radius of the cylinder; find the Mechanical Advantage.

13. If $P=8$ oz. weight, find W, when P acts at 9 inches from the axis, and the screw makes 120 turns in 8 inches.

14. If a force of 112 lbs. weight at the end of a lever 100 inches long will sustain 5 tons weight by means of a screw whose axis is vertical, and whose cylinder is 2 in. in diameter, find the inclination of the thread to the horizon.

15. If the angle of the screw be 30°, the radius of the cylinder 9 inches, the lever 4 ft. long, find P, when W is a force of $\frac{3}{4}$ ton weight.

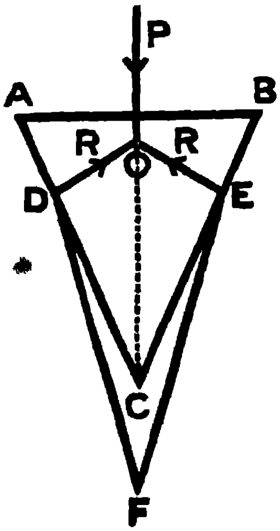
16. The circumference of the circle swept out by P is 12 ft.; how many turns must the screw make if the cylinder be 3 feet long, in order that the Mechanical Advantage may be 300?

17. If a screw make x turns in 18 inches, and the length of the power-arm is a feet, find the Mechanical Advantage of the machine.

18. The angle of a screw is 30°; if the radius of the cylinder $=r$, and the length of the power-arm $=a$, find what pressure will be caused by a force of 50 lbs. weight acting at end of the power-arm.

THE WEDGE.

254. The Wedge is a solid triangular prism made of some hard material, and is used for cleaving asunder bodies whose power of cohesion is very great.



Let ABC be a section of an isosceles Wedge, introduced into an opening DFE.

We assume that the Wedge is urged forwards by a steady pressure P, applied perpendicularly to its back AB, and that the reactions from the points of contact D, E are equal, and at right angles to the sides.

Because there is equilibrium, we may suppose the lines of action of P, and of the two resistances at D and E, to pass through a point O. (Art. 212.)

If we denote the angle of the Wedge ACB by 2α, then, resolving along OC, and at right angles to it, the Components at right angles to OC must be equal and opposite.

$$\therefore R\cos(90^\circ - EOC) = R\cos(90^\circ - DOC),$$

$$\text{or, } R\cos\alpha = R\cos\alpha.$$

And the Sum of the two Components along OC must be equal to P.

$$\therefore P = R\sin\alpha + R\sin\alpha,$$

$$\text{or, } P = 2R\sin\alpha.$$

Chisels, planes, hatchets, razors, knives are examples of this Machine.

The Wedge is driven forwards in all practical work not by steady pressure, but by a *series of blows*. This consideration shows that the action of the Wedge is most usefully treated as a *kinetic* subject.

EXAMPLES—XLIX.

1. What is the vertical angle of an isosceles wedge, when the force applied is equal to either of the two resistances?

2. A wedge is in the form of an equilateral triangle, and the two forces acting on the sides are each equal to 100 lbs. weight; find the remaining force.

3. A wedge is in the form of an isosceles triangle, and the angle of the wedge is 30°; a force of 120 lbs. weight acts opposite to this angle; find the other two forces.

255. *The Principle of Work applied to Machines.*

We shall anticipate some of the results proved in Chapter XV. on Work. It is there shown (Art. 300) that Work is measured by the product of the Force into the Distance it moves through in its own direction.

If P denote the Power, W the Weight to be overcome, and F the Friction of the parts of the machine, and if p, w, f, denote the distances through which these three forces move in the same time, then the work done by P is equal the sum of the works done by W and F, or

$$Pp = Ww + Ff.$$

If we suppose the parts of the machine to move without friction, then $F=0$, and we have—

$$Pp=Ww.$$

Now if $P<W$, then $p>w$; *i.e.* a small force can do as much work as a larger one only by moving through a proportionally greater space, and hence '*what is gained in power must be lost in time.*'

Now no machine can be made perfectly smooth, and therefore part of the work applied must be used in overcoming the resistance caused by friction; from which it follows that the *effective* work obtained from any machine is always less than the work applied to it—in other words, a machine can never create *energy*.

The ratio $\frac{\text{Useful Work}}{\text{Total Work}}$ is called the **Efficiency** of the Machine.

Example i.—Find the ratio of $P:W$ in a smooth Screw when P acts at the end of a lever.

We have $Pp=Ww$.

During one complete circuit of the power-arm, the screw advances through a distance = interval between the threads.

$$\therefore P\times\text{circumference of circle described by } P$$
$$=W\times\text{distance between the threads};$$
$$\therefore \frac{P}{W}=\frac{\text{distance between the threads}}{\text{circumference of the circle described by } P}$$

(Compare Article 253.)

Example ii.—Find the ratio of $P:W$ in a straight Lever.

In the figure of Art. 225, suppose the lever AB to turn round the fixed point F, then P and W will in a certain time describe arcs of circles.

We have $Pp=Ww$.

$$\therefore P\times\text{arc of } P\text{'s circle}=W\times\text{arc of } W\text{'s circle}.$$

Now the angles at F being equal and opposite, the arcs are proportional to the radii of the circles; $\therefore P\times AF=W\times BF$.

(Compare Article 225.)

Example iii.—Find the ratio of $P:W$ in the Wheel and Axle.

We have $Pp=Ww$.

In a single revolution the Power describes a space equal to the circumference of the Wheel, and the Weight describes a space equal to the circumference of the Axle;

$$\therefore P\times\text{circumference of wheel}=W\times\text{circumference of axle}.$$
$$\therefore \frac{P}{W}=\frac{\text{circumference of axle}}{\text{circumference of wheel}}=\frac{2\pi r}{2\pi R}=\frac{r}{R}=\frac{\text{radius of axle}}{\text{radius of wheel}}.$$

(Compare Article 235.)

Example iv.—In the First System of Pulleys (Art. 240), what power will sustain 448 lbs., if there be 4 movable pulleys?

Suppose a pulley E to rise a small distance x, then D rises $2x$, C rises $4x$, and B rises $8x$, and $\therefore$ P descends twice $8x$, or $16x$.

$$\text{We have } Pp = Ww.$$
$$\therefore P \times 16x = 448 \times x;$$
$$\therefore P = 28 \text{ lbs. weight.}$$

MISCELLANEOUS EXAMPLES—L.

1. Explain the meaning of the expression 'What is gained in power is lost in time,' and illustrate it in the case of levers of the Second and Third Orders.

2. A beam balanced about a point 7 feet from one end; when the point of support was moved 2 feet nearer that end, it was necessary to place a mass weighing 189 lbs. at that end to restore equilibrium; what was the weight of the beam?

3. A uniform rod AB, 8 inches long, is laid horizontally over two props C and D, 5 inches apart; C being 2 inches from A. Find the pressure on each prop. If a body suspended at B removes all pressure from C, show that the pressure on D is increased tenfold.

4. A uniform beam, 18 feet long, is in equilibrium on a fulcrum 2 feet from one end, when a body weighing 5 lbs. hangs at the end of the longer arm, and another weighing 110 lbs. at the other; find the weight of the beam.

5. A uniform rod, weighing 2 lbs. per foot, balances about a point 1 foot from the middle when bodies weighing 16 lbs. and 5 lbs. hang at the ends. Find the length of the rod.

6. Two men carry a weight of 112 lbs. on a light rod 7 feet long; where is it slung when one man sustains a pressure of 32 lbs. weight?

7. A uniform straight rod, 5 feet long and weighing 168 lbs., rests on a prop at each end; where must a body weighing 280 lbs. be slung that the pressures on the props may be in the ratio 3 to 2.

8. A bar 6 feet long, and weighing 50 lbs., rests on two props distant 1 foot and 2 feet from the ends; what weight acting at one end will make the pressures equal on the two props?

9. A uniform bar, weighing 6 oz. per foot, balances about a point 16 inches from one end when a body weighing 12 lbs. hangs at that end; find the length of the bar.

10. If a balance have equal arms, but unequal scale-pans, show that the true weight of a body is half the sum of its apparent weights in the two scale-pans.

11. A steelyard is 48 inches long and weighs 2 lbs.; its C. G. is 6 inches from the end at which scale-pan hangs, and the shorter arm is 4 inches. Find (1) the distance between the graduations, (2) the greatest weight which the instrument can determine.

12. A pair of nut-crackers is 5 inches long, and a pressure of $3\frac{1}{2}$ lbs. weight at the end will crack a nut placed $\frac{7}{8}$ inch from the hinge; what mass placed on the nut will crack it?

13. Two masses, m and m_1, balance on a wheel and axle. If they be interchanged, find the acceleration of the body suspended from the wheel, the mass of the machine being neglected.

14. If in a wheel and axle $R=3r$, and $P+W=48$ lbs. weight, find P and W.

15. In a false balance a body whose true weight is A has an apparent weight B, and one whose true weight is C appears to weigh D; prove that the true weight of a body which appears to weigh E is

$$\frac{E(A-C)+CB-AD}{B-D}.$$

16 A wheel whose weight is W and radius r meets an obstacle the height of which is x; what horizontal force acting at the centre will cause the wheel to surmount the obstacle?

17. A cone whose vertical angle is a, and whose weight is W, fits into an opening in a table cut out in the form of an equilateral triangle; find the pressure at each point of contact.

18. Two spheres whose radii are 3 inches and 5 inches rest against each other, being suspended from the same point by strings 3 inches and 5 inches respectively; if the shorter string makes an angle of 45° with the vertical, compare the weights of the spheres.

19. In a steelyard the graduated arm is a foot long, and pounds are measured by a movable weight of x oz., and stones by an addition weight of y oz. at the end of the arm; find the distance between the graduations. (*Note.*—When complete stones are in the pan the movable weight is at the zero.)

20. In the Third System there are n pulleys each of weight x. The free end is passed under a movable pulley, then over a fixed pulley, and then made fast to the light bar carrying the weight W. Find W in terms of x and n.

21. A string made fast at one end passes under a movable pulley whose mass is m_1, and over a fixed pulley, and to the free end a mass m is attached; find the tension of the string.

22. In the Second System there are n strings at the lower block, and the mass at the free end of the strings is m times that which would produce equilibrium; find the acceleration of the ascending body.

23. In the First System there are two movable pulleys, and a weight of 128 lbs. is supported by a body weighing 32 lbs. hanging at the free end of the string; what mass must be added to the 32 lbs. so that it may descend with an acceleration of $\frac{1}{2}g$?

24. In the same system the two pulleys are of equal mass. A mass of 100 lbs. is at rest when a mass of 28 lbs. is fastened to the free end. How much must be taken from the 100 lbs. to cause the 28 lbs. to descend with an acceleration of $\frac{3}{4}g$?

25. To one end of a string passing over a smooth peg is attached a mass of 30 lbs., to the other a light pulley; over the pulley is passed a string carrying at its ends masses of 9 lbs. and 7 lbs. respectively; find the acceleration of each body and the tensions.

26. If the pulley in Ex. 25 had a mass of 2 lbs., find the accelerations and the tensions.

27. If in Ex. 25 the 7 lbs. were replaced by a body whose mass is m lbs., find the value of m, so that the body should have no motion in fixed space.

28. Two equal spheres of 1 inch radius are in equilibrium in a spherical cup whose radius is 3 inches; show that the pressure between the cup and either sphere is double the pressure between the two spheres.

29. A heavy sphere rests on two parallel rods in the same horizontal plane and at a distance from each other equal to the radius of the sphere; find the pressure on either rod.

30. A smooth sphere whose weight is W rests upon two planes of equal inclination (θ) which are placed on a smooth horizontal table, and are prevented from sliding by a horizontal string which ties them together; find the tension of the string.

31. Three equal smooth spheres (each of weight W) are placed in contact on a smooth table, and another sphere (equal in all respects to the others) is then placed on them. What horizontal force applied to each of the lower spheres will prevent their separating?

32. Two spheres whose radii are 3 feet and 5½ feet and weights are W_1 and W_2 respectively, are placed between two vertical rods 16½ feet apart, the larger sphere resting on the ground, the centres being in the vertical plane of the rods; find all the reactions.

33. A smooth hemispherical bowl whose base is closed is placed with the base vertical. Inside are two equal spheres, each of weight $20\sqrt{3}$ lbs., their radii being one-third the radius of the bowl; find the pressures at all points of contact.

34. A light string having a mass m at one end passes over a movable pulley A whose mass is m_1, under another movable pulley B whose mass is m_2, then over a fixed pulley, and the other end is made fast to A. All the strings being vertical, find the acceleration of each body, and the tension of the string.

35. If in Ex. 34 $m=8$ lbs., $m_1=2$ lbs., $m_2=6$ lbs., how far does m move in 5 seconds.

36. If in Ex. 34 $m_2=10$ lbs., $m_1=1$ lb., what must be the value of m that it may remain stationary?

CHAPTER XII.

FRICTION.

256. SURFACES are said to be *smooth* when no resistance is caused by them to the motion of one body over another in contact with it. When there is such resistance they are said to be *rough*. The resistance so caused is termed **Friction.**

Our knowledge of the Laws of Friction is derived principally from the experiments made in 1781 by Coulomb, a French officer of Engineers. Further experiments carried out by General Morin in 1831-34 confirmed Coulomb's results. Recent investigations have shown, however, that some of these Laws are not rigorously correct, and that in some circumstances Law 5 in the next Article is seriously in error.

257. The **Laws of Friction** are as follows:—

1. The amount of Friction depends on the material of which the bodies are composed.
2. For the same pair of substances the Friction varies directly as the normal pressure.
3. For the same pressure the Friction is independent of the extent of the surfaces in contact.

(There must be a definite area of contact—*i.e.* a physical line and point of contact are excluded.)

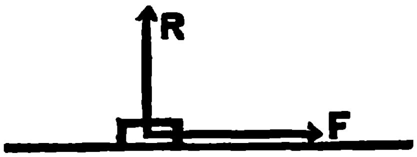
4. The Friction increases with the time of contact up to a certain period, after which it becomes constant.
5. The Friction is independent of the velocity when one body is sliding on the other.
6. Rolling Friction is always less than Sliding Friction.

258. Unless there is some tendency of one body to slide over the surface of another, the force of Friction is nonexistent. More than a certain amount of friction can never be called into play. The greatest amount is exerted when one body is *just on the point of motion* over the other. This amount is called the **Limiting Friction**; it is also known as **Statical Friction.**

When one body has motion over another, the friction called into action is called **Kinetic Friction.** It is always somewhat less than the Limiting Friction.

259. The value of the Limiting Friction bears to the normal pressure a constant ratio, the value of which depends on the nature of the surfaces in contact. This ratio is called the **Coefficient of Friction**; it is generally denoted by μ.

Thus, if a body be *on the point of moving* under the action of the tangential force F, and R be the normal pressure between the surfaces, then



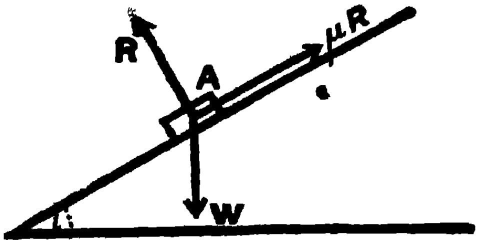
$$\frac{F}{R}=\mu, \text{ and } \therefore F=\mu R.$$

In new wood, even when planed, μ is about $\frac{1}{2}$; in metals it is about $\frac{1}{4}$; in wood and metal about $\frac{2}{5}$; in leather on wood (wetted with water) nearly $\frac{8}{10}$. When the surfaces of wood are worn by long rubbing, μ is reduced to about $\frac{1}{3}$.

260. Def.—*The* **Angle of Friction** *is the greatest inclination of the rough plane on which a body is just on the point of sliding when under the action of its weight and friction only.*

261. Theorem.—*The Tangent of the Angle of Friction is equal to the Coefficient of Friction.*

Let a body, A, be placed on a plane inclined at such an angle that the body is just on the point of slipping down. Then the forces acting on A are (1) its Weight, (2) the Pressure of the plane, (3) the Friction up the plane.



Resolving along the plane, $\mu R = W \sin i$ . . . . . . (1.)

" at right angles, $R = W \cos i$ . . . . . . (2.)

Divide (1) by (2), $\therefore \mu = \tan i$. (Q.E.D.)

262. We may therefore determine the value of μ for any pair of substances as follows:—Place one body on the other, and cant the latter until the former is on the point of slipping. The tangent of the angle of inclination at this instant will be the Coefficient of Friction for those substances.

263. The student will observe that Friction always *opposes motion.* Hence if a body is sliding down a plane, the Friction acts up the plane; if the body is moving up the plane the Friction acts down.

264. *To find the Acceleration on a rough inclined plane.*

Suppose the body to be moving *down*, then the Friction is acting up the plane.

The Force causing the motion $= mg \sin i - \mu R$. (See last diagram.) Let a be the acceleration produced.

Now, $ma =$ Force causing motion;

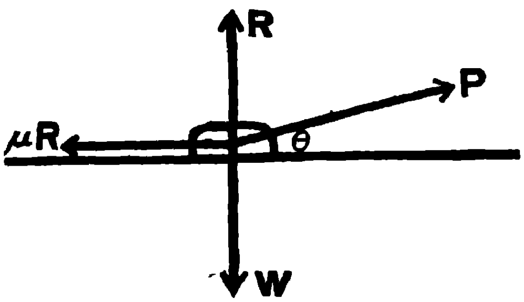
$$\therefore \quad ma = mg \sin i - \mu . mg \cos i;$$

$$\therefore \quad a = g(\sin i - \mu . \cos i).$$

If the body were projected *up* the plane, the retardation to the motion, in like manner, may be shown to be

$$a = g(\sin i + \mu . \cos i).$$

265. *To find the Magnitude and Direction of the Least Force which will draw a body along a rough horizontal plane.*



Resolve along the plane;

$\therefore P \cos \theta = \mu R$ . . . (1.)

Resolve at right angles to the plane;

$\therefore P \sin \theta + R = W$. (2.)

Eliminate R from these equations, and we obtain—

$$P = \frac{\mu . W}{\cos \theta + \mu \sin \theta}.$$

Let ϵ be the 'Angle of Friction,' then $\tan \epsilon = \mu$. (Art. 261.)

$$\therefore P = \frac{W \tan \epsilon}{\cos \theta + \tan \epsilon . \sin \theta}.$$

From which equation, $P = \dfrac{W \sin \epsilon}{\cos (\theta - \epsilon)}$.

To make P least, we must have $\cos (\theta - \epsilon)$ greatest, *i.e.* $= 1$;

$$\therefore \theta - \epsilon = 0;$$

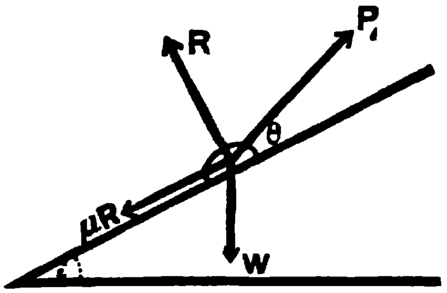
$$\therefore \theta = \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$\text{Then } P = W \sin \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Equation (1) gives the Direction, and (2) gives the Magnitude of the Least Force of Traction. Therefore when the direction of the Force of Traction makes an angle with the plane equal to the 'Angle of Friction,' the body will be moved by the least force of Traction.

266. *A body is kept in equilibrium on a rough inclined plane by a force acting in a given direction; to find the Limits between which the magnitude of the force must lie.*

First, let P_1 be the magnitude of the force which is on the point of causing the body to move *up* the plane,



Then the friction acts *down.*

Resolving along the plane and at right angles to it—

$$P_1 \cos \theta = \mu R + W \sin i \quad . \quad . \quad . \quad (1.)$$

$$P_1 \sin \theta + R = W \cos i \quad . \quad . \quad . \quad . \quad (2.)$$

Eliminate R from these equations, and we obtain

$$P_1 = W . \frac{\sin i + \mu \cos i}{\cos \theta + \mu \sin \theta}.$$

Next, let P_2 be the magnitude of the force which is on the point of being overcome by the body moving *down* the plane. Then the friction acts *up.*

Resolving as before, we have—

$$P_2 \cos \theta + \mu R = W \sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$P_2 \sin \theta + R = W \cos i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Eliminate R from these equations, and we obtain

$$P_2 = W.\frac{\sin i - \mu \cos i}{\cos \theta - \mu \sin \theta}.$$

From these two results we conclude that if P has any value between P_1 and P_2, that is, between

$$W.\frac{\sin i + \mu \cos i}{\cos \theta + \mu \sin \theta}, \text{ and } W.\frac{\sin i - \mu \cos i}{\cos \theta - \mu \sin \theta},$$

the body will be in equilibrium on the plane.

If the force act along the plane, then $\theta = 0$, and $P_1 = W(\sin i + \mu \cos i)$; $P_2 = W(\sin i - \mu \cos i)$.

NOTE.—P_2 can be deduced from P_1 *by changing sign of* μ. Similarly in other cases, when two limits must be found.

The Screw with Friction.

267. Let a force P_1 applied horizontally at the circumference of the cylinder be on the point of overcoming the Resistance ($= W$) and the Friction ($= \mu R$).

Resolving along the plane and at right angles to it—

$$P_1 \cos i = \mu R + W \sin i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$P_1 \sin i + W \cos i = R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Eliminate R from these equations, and we obtain

$$\frac{P_1}{W} = \frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}.$$

Let the power P act at the end of an arm whose length is R, and let the radius of the cylinder be r.

$$\text{Then, } \frac{P_1}{P} = \frac{R}{r}, \text{ or } P_1 = \frac{P.R}{r};$$

$$\therefore \frac{P}{W} = \frac{r}{R} \cdot \frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}.$$

If W is on the point of overcoming the combined effect of P_2 and the Friction, it may be shown that

$$\frac{P}{W} = \frac{r}{R} \cdot \frac{\sin i - \mu \cos i}{\cos i + \mu \sin i}.$$

From these results we conclude that there will be equilibrium if $\frac{P}{W}$ lies between $\frac{r}{R}\cdot\frac{\sin i+\mu\cos i}{\cos i-\mu\sin i}$ and $\frac{r}{R}\cdot\frac{\sin i-\mu\cos i}{\cos i+\mu\sin i}$

If we write $\tan\epsilon$ for μ, these limits may be written

$$\frac{r}{R}\cdot\tan(i+\epsilon) \text{ and } \frac{r}{R}\tan(i-\epsilon).$$

Example i.—A mass of 40 lbs. rests in a state bordering on motion on a plane inclined to the horizon at an angle of 30°; find the coefficient of friction.

We may use the theorem of Art. 261, and say at once that $\mu=\tan 30°$;

$$\therefore\ \mu=\frac{1}{\sqrt{3}},$$

Or we may proceed, as in that investigation, as follows:—

Let W be the weight of the body.

Resolving along the plane, and at right angles to it—

$$\mu R=W\sin 30° \quad \ldots\ldots \quad (1.)$$
$$R=W\cos 30° \quad \ldots\ldots \quad (2.)$$
$$\therefore\ \mu=\tan 30°=\frac{1}{\sqrt{3}}.$$

Example ii.—What force, acting along the plane, will draw a body whose weight is 100 lbs. along a rough horizontal plane. ($\mu=\frac{1}{2}$.)

$$F=\mu R=\tfrac{1}{2}\times 100=50.$$

Hence a force of 50 lbs. weight will cause a state bordering on motion.

Example iii.—A force of 30 lbs. weight acting parallel to the plane just supports a body whose weight is 50 lbs. on a smooth inclined plane. If the plane were rough, and $\mu=\frac{1}{10}$, find the force which, acting along the plane, would draw the body up.

The friction acts *down*. Let P be the force.

Resolving along the plane, and at right angles to it—

$$P=50\sin i+\mu R \quad \ldots\ldots \quad (1.)$$
$$R=50\cos i.$$

From which equation we obtain—

$$P=50\sin i+\mu\ 50\cos i.$$

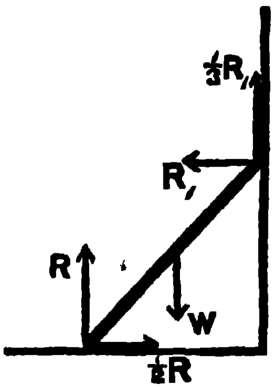
Now, from the first part of the question it may be shown that—

$$\sin i=\tfrac{3}{5};\ \therefore\ \cos i=\tfrac{4}{5},\text{ and } \mu=\tfrac{1}{10}.$$

Substituting these values, we find that $P=34$.

Therefore, any force exceeding 34 lbs. weight will draw the body up the plane.

Example iv.—A ladder, whose C. G. is at its middle point, rests with one end on a horizontal plane, and the other against a vertical wall, to which it is inclined at an angle of 45°. The coefficient of friction for the lower end $=\frac{1}{2}$, and for the upper end $=\frac{1}{3}$. A man whose weight is half that of the ladder mounts it. How far will he ascend before the ladder slips?



Let a = length of ladder;

x = distance ascended by the man.

We use the conditions of equilibrium obtained in Article 218.

Equating the vertical components, we have—

$$R+\tfrac{1}{3}R_1=W+\frac{W}{2}\ \ .\ .\ .\ .\ (1.)$$

Equating the horizontal components, we have—

$$\tfrac{1}{2}R=R_1\ \ .\ .\ .\ .\ (2.)$$

Equating moments round the lower end, we have—

$$W.\ \tfrac{1}{2}\,a\cos 45^\circ+\tfrac{1}{2}W.\ x\cos 45^\circ=R_1\,a\sin 45^\circ+\tfrac{1}{3}R_1\,a\cos 45^\circ\ .\ .\ .\ (3.)$$

Solving these equations we shall obtain $x=\frac{5}{7}a$.

Examples—LI.

1. Find the coefficient of friction if a body whose weight is W just rests on a rough plane inclined at an angle of 45° to the horizon.

2. A body whose weight is 24 lbs. just rests on a rough plane inclined at an angle of 30° to the horizon; find μ, and the value of the friction called into play.

3. If the tangent of inclination of a rough inclined plane be $\frac{3}{8}$, and a body whose weight is 24 lbs. be just supported by the friction alone, find the least force which, acting along the plane, will draw the body up the plane.

4. A plane rises 5 in 13; find the weight of the greatest mass which can just be retained at rest by a force equal to the weight of 20 lbs. acting along the plane. ($\mu=\frac{1}{4}$.)

5. A horizontal force of 11 lbs. weight is just on the point of moving a body whose weight is 13 lbs. up a plane, the tangent of the inclination being $\frac{1}{3}$; find μ.

6. At what inclination must a wooden slip be laid so that blocks of iron may slide down by the action of gravity? ($\mu=\cdot 625$.)

7. A body, whose weight is 100 lbs., is just kept at rest by friction on a rough inclined plane. The height of the plane is 5 feet, and its length is 25 feet. What is the force and coefficient of the friction?

8. State the laws of statical friction, and explain what is meant by the *Angle of Friction.*

9. Define the terms: *Friction, Coefficient of Friction, Limiting Friction.*

10. How may the Coefficient of Friction be determined by experiment?

11. Find the least force which will drag a body, whose weight is 50 lbs., along a rough horizontal plane, the coefficient of friction being $\frac{1}{\sqrt{3}}$

12. At what angle of inclination (θ) should the traces be attached to a sledge that it may be drawn up a hill, whose gradient is known, with the least possible exertion?

13. Prove that the Resultant of the Normal Pressure and the Limiting Friction is always inclined to the normal at an angle equal to the Angle of Friction. *Note.*—This is very important.

14. A body, whose weight is 50 lbs., rests by friction only on a plane whose inclination is 30°; find the force of friction in action, and also the pressure exerted on the plane.

15. A heavy body just rests on a rough inclined plane. Show that the *total resistance* due to the plane is $R \sec \epsilon$ acting at an angle ϵ with the normal, when R is the normal pressure, and ϵ is the angle of friction.

16. Determine the magnitude of the limiting angle of friction, having given the coefficient of friction.

17. If the roughness of a plane, which is inclined to the horizon at a known angle i, be such that a body whose weight is W will just rest upon it, find the *least* force required to draw the body up.

18. A force of 30 lbs. weight just supports a body whose weight is 50 lbs. on a smooth inclined plane. If the plane were rough, and the coefficient of friction $\frac{1}{10}$, what force acting parallel to the plane will just keep the body from moving down the plane?

19. A force acting at an angle of 60° with the horizon can just support a body of certain weight on a rough plane inclined to the horizon at an angle of 30°. Were the plane smooth, the same force acting parallel to the plane would preserve equilibrium. Find the coefficient of friction.

20. The least force that will draw a body whose weight is 112 lbs. along a rough horizontal plane is a force of 16 lbs. weight; find the least force that will draw it up an equally rough plane inclined to the horizon at an angle of 60°.

21. If a body be kept at rest on an inclined plane by a force making a given angle with the plane, show that the reaction of the plane when it is smooth is the Harmonic Mean between the greatest and least reactions when it is rough.

22. If a body whose weight is W lbs. rest on a plane, the tangent of the inclination of which is $1/x$, and if the force of the friction be the weight of 1 lb. in m lbs. weight of the body, find the least force which, acting parallel to the plane, will be on the point of causing the body to move up.

23. A body whose mass is $\sqrt{2}$ lbs. is placed on a rough plane inclined to the horizon at an angle of 45°. If $\mu=\dfrac{1}{\sqrt{3}}$, find in what direction a force equal to the weight of $(\sqrt{3}-1)$ lbs. must act on the body in order to support it.

24. If a force in a given direction keep a body at rest on a rough plane, find the limits between which the force must lie.

25. If a force equal to the weight of $\sqrt{3}$ lbs., acting along a plane, would just drag a body whose weight is 2 lbs. up the plane, while a force equal to the weight of 1 lb. would just prevent it slipping down, find the amount of friction exerted when a force equal to the weight of $\sqrt{2}$ lbs. acting up the plane keeps the body in equilibrium.

26. Find the magnitude and direction of the *least* force (1) which will just draw a weight of 50 lbs. up a rough plane inclined at 60° to the horizon, and (2) which will just prevent its slipping down; the coefficient of friction being $\dfrac{1}{\sqrt{3}}$.

27. A weight W is just supported by friction on a plane inclined at an angle α to the horizon. Show that it cannot be moved up the plane by any horizontal force less than $W\tan 2\alpha$.

Explain this result when $\alpha=\dfrac{\pi}{4}$.

28. A body whose weight is 100 lbs. is kept in equilibrium on a rough plane, whose inclination is i, by a force P inclined at an angle θ to the plane; if the greatest angle at which the body would rest by friction alone is 45°, find the limits between which P must lie.

29. Investigate the condition of equilibrium, when a body of weight W is supported on a rough plane inclined to the horizon at an angle α, by a force P, inclined to the plane at an angle β; the coefficient of friction being μ, and the body being on the point of moving down the plane.

30. If the ratio of the greatest to the least force which, acting parallel to a rough inclined plane, can support a weight on it, be equal to the ratio of the weight to the pressure on the plane, show that the coefficient of friction will be $\tan\alpha . \tan^2\frac{\alpha}{2}$, when α is the inclination of the plane to the horizon.

31. What is the least coefficient of friction which will allow of a heavy body being just kept from sliding down an inclined plane of a given inclination, the body (whose weight is W) being just sustained by a given horizontal force P?

32. An isosceles triangle whose base is to a side as $1 : \sqrt{7}$ is placed with its base on an inclined plane; and it is found that when it begins to slide it at once topples over. Find the coefficient of friction.

33. A cone is placed on a rough plane, the coefficient of friction being $\tan\epsilon$; find its vertical angle if the cone begins to slip and topple over simultaneously.

34. A parallelopiped rests on an inclined plane. Its base is 2 feet square. What is its height if it begin to slide and topple over at the same moment? ($\mu=\frac{1}{2}$.)

35. Two bodies, whose weight are W_1 and W_2, rest on a rough narrow inclined plane and are connected by a string; if the coefficients of friction are μ_1, μ_2 respectively, find the greatest inclination (i) of the plane which is consistent with equilibrium.

36. Investigate the conditions of equilibrium when a power P supports a body, whose weight is W, on a rough screw, μ being the coefficient of friction.

37. The distance between the threads of a screw is such that a body is just sustained without the action of any power; find the least power which will overcome a given weight W, the coefficient of friction being μ.

38. If $\mu=\frac{1}{2}$, and the length of the screw be 2 feet, its circumference being 6 inches, find the least number of turns which may be given to its thread in order that a body of given weight may be supported upon it without the action of any power.

39. Show that with a given angle of friction, the power necessary to raise a body of given weight increases indefinitely as the angle of the screw approaches a certain value. What happens if the angle exceed this value?

40. A screw is worked by a lever, the end of which describes a circle of 12 feet. The circumference of the screw is 6 inches, and the interval between the threads is 1 inch. If $\mu=\frac{1}{4}$, find the pressure exerted by a force of 20 lbs. weight at the end of the lever.

41. The diameter of a screw press is 10 feet, the interval between the threads is 1 inch, the diameter of the screw is 9 inches, and $\mu=\frac{1}{3}$. What pressure is exerted by a power of 10 lbs. weight?

42. The limiting position of equilibrium of a ladder resting against a wall equally rough with the ground is at an angle of 45°; find the value of μ.

43. A uniform ladder 10 feet long rests with one end against a smooth vertical wall and the other end on the ground ($\mu=\frac{1}{2}$); find how high a man whose weight is four times that of the ladder may ascend before the ladder begins to slip, the foot of the ladder being 6 feet from the wall.

44. A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the C. G. of the ladder being one-third of the length from the foot. The coefficients of friction at the top and bottom are $\frac{1}{4}$ and $\frac{1}{3}$ respectively. How high on the ladder may a man, whose weight is half the weight of the ladder, ascend before the ladder slips?

45. A ladder whose length is x and weight W has one foot on the ground and the other against a vertical wall, the coefficients of friction being μ and μ_1 respectively. If the distance of the C. G. from the foot be y feet, determine the inclination (θ) of the ladder when on the point of slipping.

46. A uniform beam whose weight is W leans against a vertical wall, and has its lower end resting on a horizontal plane. If μ and μ_1 be the coefficients at the top and bottom respectively, find the inclination (θ) of the beam to the horizon when the beam is on the point of motion; find also the pressures on the wall and the plane.

47. If the beam in the last question be not uniform, find the value of θ, when the distances of the C. G. from the top and bottom are y and x respectively.

48. Find the limiting positions of equilibrium of a beam resting on two equally rough inclined planes, having given the inclinations of the planes $=i$ and i_1, the segments of the beam made by the C. G. $=x$ and y; the angle of friction $=\epsilon$; and the inclination of the beam to the horizon $=\theta$.

49. One end of a uniform heavy rod rests on a smooth inclined plane whose angle is 30°, and the other rests against a rough vertical wall which passes through the foot of the inclined plane. If the coefficient of friction $=\mu$, find the angle made by the rod with the vertical in the limiting position of equilibrium.

50. If the ladder in Ex. 45 be placed in a known position a, at what distance (a) from the foot must a body whose weight is W_1 be placed so that the ladder may be on the point of slipping? ($\mu=\mu_1$.)

51. AB is a string of length a, of which one end is fixed at A, while the other is attached to an end B of a uniform rod BC also of length a; and of this rod the other extremity rests on a rough horizontal plane at a distance a below A. Supposing that the end of the rod is about to slip when C is vertically below A, find the coefficient of friction.

52. A uniform rod rests with one extremity against a rough vertical wall ($\mu=\frac{7}{8}$), the other extremity being supported by a string three times the length of the rod attached to a point in the wall; show that the angle the string makes with the wall in the limiting positions of equilibrium is $\tan^{-1}(\frac{5}{27})$ or $\tan^{-1}(\frac{1}{3})$.

53. If two bodies whose weights are W_1 and W_2 be placed on two equally rough planes having a common vertex, and be connected by a string over this common vertex, show that if the inclinations be i and i_1 respectively, $W_1 \sin(i-\epsilon)=W_2 \sin(\epsilon+i_1)$, where $\tan \epsilon=\mu$.

54. A body, whose weight is W, can be just sustained on a rough inclined plane by a force P_1 acting horizontally, or by a force P_2 acting along the plane; find the value of the angle of friction.

55. A heavy particle, weight W, just rests in equilibrium on a rough inclined plane; if now it be further acted on by a force $=W$, within what limits must the direction of this force lie in order that the equilibrium may not be disturbed?

56. A uniform rod rests with its upper end against a smooth vertical plane, and its lower end upon a rough inclined plane. When the rod is on the point of slipping, show that its inclination to the horizon is $\tan^{-1}\left\{\frac{\cot(\epsilon-a)}{2}\right\}$, where a is the inclination of the plane.

57. Two particles, connected by a string, lie one on each slope of a double inclined plane of inclination a; one plane is rough, the other smooth. If there be motion, the ratio of the weight of the body on the smooth plane to the weight of that on the rough plane must not lie

between $1-\mu \cot \alpha$ and $1+\mu \cot \alpha$, when μ (less than $\tan \alpha$) is the coefficient of friction.

58. What force will pull a body whose mass is 12 cwt. along a rough horizontal plane ($\mu=\frac{1}{4}$), (1) with uniform velocity of x f.-s.; (2) with a uniform acceleration of y f.-s.-s? (Force acts horizontally.)

59. A hemisphere rests with its curved surface touching a vertical wall ($\mu_1=\frac{8}{19}$) and a horizontal plane ($\mu_2=\frac{1}{8}$). Show that when the body is on the point of slipping, the inclination of the plane of its base to the horizon is $\sin^{-1}$ ($\frac{9}{20}$). *Note.*—If the radius of the sphere is r, the C. G. of its hemisphere is at a distance $\frac{3}{8}r$ from the centre of the sphere.

60. The difference between two like forces, acting parallel to the plane, which will keep a body whose weight is 10 lbs. in limiting equilibrium on a plane inclined at an angle of 45° to the horizon is 8·16 lbs.; show that the angle of friction is nearly 30°.

61. A uniform beam of length $2l$ is placed on a rough peg, and rests with its lower extremity against a rough vertical wall. If the coefficient of friction between the beam and the peg, and also between the beam and the wall, be $\frac{1}{2}$, and the beam be on the point of sliding down between the peg and the wall when its inclination is 45°, find the distance of the peg from the wall.

62. A uniform rod is freely movable round its middle point fixed at the summit of an inclined plane. A body whose weight is P hangs from one end of the rod, and to the other end a string is fastened, of length equal to half that of the rod. To this a body, whose weight is W, is attached, which rests on an inclined plane whose inclination is α. If the plane be smooth, and there be equilibrium when the rod is horizontal, prove that $P=2W\sin^2 \alpha$.

If the plane be rough, and the friction such that W would just rest on it without support, prove that $W\sin^2 2\alpha$ is the greatest value P can have without disturbing the equilibrium.

63. A body of mass m_1 is placed on a rough horizontal table, and is connected by a string over its edge with a mass m which descends; find the acceleration, and the tension of the string.

64. In the second System of Pulleys $W=4$ lbs. and $P=1$ lb., the Mechanical Advantage being 4. Prove that if a rope loses one-tenth of its tension in passing round a pulley in motion, then about 0·29 lb. wt. must be added to the 1 lb. wt. in order to raise the 4 lbs.

CHAPTER XIII.

IMPACT.

268. Impact is the term used to denote the action which lasts for an extremely short time when bodies come into collision.

269. When one body strikes another moving in the same direction, it is a matter of common experience that one loses velocity and the other gains velocity. If we suppose that the bodies separate after collision, then although the bodies are in contact for only a very short time, yet that time is sufficiently long for the pressure between them to increase from zero at the instant of collision to its maximum, and then to decrease to zero at the instant of separation.

By the Third Law of Motion this pressure must act equally on the two bodies. By the Second Law it will produce an equal change of momentum in each body. Therefore the momentum generated in one must be equal and opposite to that generated in the other; and hence we infer that the momentum (*reckoned in the positive direction*) lost by one body has been gained by the other; in other words, that the ***whole momentum of the system after impact is equal to the whole momentum before impact.***

Notice must be taken of the words in brackets; they point out that the whole momentum means the algebraical sum of the momenta, and thus the case is included where the bodies are moving in opposite directions.

270. *When one body overtakes another moving in the same direction, to find the velocity after impact if the bodies keep together.*

Let a body whose mass $=m$, moving with velocity $=u$, impinge on another whose mass $=m_1$, moving with velocity $=u_1$.

Let $v=$ velocity after impact.

Now, Momentum before impact = Momentum after;

$$\therefore\ mu+m_1u_1=(m+m_1)v.$$

$$\therefore\ v=\frac{mu+m_1u_1}{m+m_1}.$$

If the bodies *meet*, then we must change the sign of either u or u_1.

The Student is recommended to pay no direct attention to this result when a question is proposed, but to apply the Third Law of Motion directly, as in the following examples.

Example i.—A ball whose mass is 10 lbs., moving with a velocity of 12 f.-s., overtakes another ball whose mass is 6 lbs. moving with a velocity of 4 f.-s.; find the common velocity after impact.

Let $v=$ velocity after impact.

Momentum before = Momentum after.

$$\therefore\ 10\times 12+6\times 4=(10+6)\,v;$$

$$\therefore\ 120+24=16\,v;$$

$$\therefore\ v=9 \text{ f.-s.}$$

Example ii.—If the balls meet, find the common velocity after impact.

Suppose the velocity 4 f.-s. to be in the negative direction.

Let $v=$ velocity required;

Momentum before = Momentum after;

$$\therefore\ 10\times 12-6\times 4=(10+6)\,v;$$

$$\therefore\ 120-24=16\,v;$$

$\therefore\ v=6$ f.-s. in the direction of the larger mass.

EXAMPLES—LII.

The bodies keep together after impact.

1. A mass of 20 lbs., moving with a velocity of 40 f.-s., *overtakes* a mass of 60 lbs. moving with a velocity of 30 f.-s.; find the velocity after striking.

2. If the masses in Ex. 1 *meet*, find the velocity after striking.

3. A mass of 3 lbs., with velocity of 20 f.-s., overtakes a mass of 10 lbs. moving with a velocity of 12 f.-s.; find the velocity after impact.

4. If the bodies in Ex. 3 *meet*, find the velocity after striking.

5. Two equal masses are moving in the *same* direction with velocities of 15 and 10; find the velocity after impact.

6. If the same bodies were moving in *opposite* directions with velocities of 40 and 32 respectively, find the velocity after impact.

7. Two masses in the ratio 2 : 3 are moving (1) in the *same* directions with velocities 25 and 20; (2) in the *opposite* directions with those velocities. Compare the velocities after impact in the two cases.

8. A mass of 10 lbs., moving with a velocity of 20 f.-s., strikes a mass of 50 lbs. at rest; find the velocity after striking.

9. A mass moving with a velocity of 100 f.-s. strikes a body of three times the mass at rest; find the velocity after impact.

10. A mass of 15 lbs., moving with a velocity of 25, overtakes a mass of 35 lbs.; the common velocity after the collision is 20; find the velocity with which the larger mass was moving before impact.

11. Two masses, m and $\frac{2}{3}m$, moving with velocities v and $\frac{4}{5}v$ respectively in the same direction, came into collision; find the velocity lost by one and gained by the other.

12. There are five bodies whose masses are 1, 3, 5, 7, 9 lbs. respectively, arranged in a smooth groove. The first impinges on the second with a velocity of 400 f.-s.; these two impinge on the third, and so on; find the final velocity when all are in motion.

271. We have, up to the present, considered that when one body impinges on another it does not separate from it, so that after the collision the two must be regarded as forming a single body.

Now, all bodies when compressed have a greater or less tendency to return to their original forms on the removal of the compressing force. The internal force which a body thus exerts to recover its original form is called the **Force of Restitution.**

This force, like all others, is measured by the amount of momentum which it is capable of producing.

272. It is matter of experiment that for each pair of substances, whatever be their masses, and whatever be their velocities before impact, their relative velocity after impact bears a *constant ratio* to their relative velocity before the collision. This constant ratio is known by the various names, 'Coefficient of Elasticity,' 'Index of Elasticity,' 'Modulus of Elasticity,' 'Coefficient of Restitution.' It is always denoted by the letter e.

If u and u_1 be the velocities before impact,
and v and v_1 „ „ after impact;
Then $u-u_1$ is the relative velocity before impact,
and v_1-v is the magnitude of the relative velocity after.

$$\text{Hence } \frac{v_1-v}{u-u_1}=e\,; \quad \therefore\ v-v_1=-e\,(u-u_1),$$

where e has to be determined for each pair of substances.*

If $e=1$, the bodies are said to be *perfectly elastic.*
If $e<1$, „ „ „ *imperfectly elastic.*
If $e=0$, „ „ „ *perfectly inelastic.*

We shall suppose that the bodies are spheres.

273. The line joining the centres of two spheres at the instant of impact is called the *line of impact.* When the centres of the spheres are moving in the line of impact the impact is said to be *direct;* if not, the impact is said to be *oblique.*

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* When two balls of cast-iron impinge, $e=\cdot 66$.
„ „ glass „ $e=\cdot 94$.
„ „ ivory „ $e=\cdot 81$.
When a cork ball impinges on an ivory ball, $e=\cdot 60$.
When a cast-iron ball „ „ lead ball, $e=\cdot 13$.

DIRECT IMPACT OF TWO SPHERES.

274. Let a body, whose mass is m moving with a velocity u, impinge on a body whose mass is m_1 moving with a velocity u_1 in the same direction; to find the velocities after impact.

Let the velocities after impact be v and v_1 respectively;

$$\text{Then, } v - v_1 = -e\,(u - u_1) \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1.)$$

$$\text{and, } mu + m_1u_1 = mv + m_1v_1 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2.)$$

On solving these equations (1) and (2), we obtain—

$$v = \frac{mu + m_1u_1 - em_1\,(u - u_1)}{m + m_1};$$

$$\text{and } v_1 = \frac{mu + m_1u_1 + em\,(u - u_1)}{m + m_1}.$$

NOTE.—If the bodies are moving in *opposite* directions, we must consider one of the velocities as negative.

COR. I.—If $m = m_1$, and $e = 1$, then these results become

$$v = u_1; \quad v_1 = u.$$

And we infer that if two equal and perfectly elastic balls impinge directly, they will *exchange their velocities.*

COR. II.—If $e = 0$, then $v = \dfrac{mu + m_1u_1}{m + m_1}$; and $v_1 = \dfrac{mu + m_1u_1}{m + m_1}$.

And this is the result obtained in Article 270, Ex. v.

Example i.—A mass of 5 lbs., moving with a velocity of 14 f.-s., impinges on a mass of 3 lbs. moving in the same direction with a velocity of 8 f.-s.; find the velocities after impact, if $e = \frac{1}{3}$.

Let v = velocity of the 5 lbs.; v_1 = velocity of the 3 lbs.

$$\text{Then } 5 \times 14 + 3 \times 8 = 5v + 3v_1 \quad . \ . \ . \ . \ . \ . \ . \quad (1.)$$

$$\text{and } v - v_1 = -\tfrac{1}{3}\,(14 - 8) \quad . \ . \ . \ . \ . \ . \quad (2.)$$

From which equations $v = 11$; and $v_1 = 13$.

NOTE.—The student will notice that the 5 lbs. mass loses a Momentum 15; and the 3 lbs. mass gains a Momentum 15.

Example ii.—A mass of 20 lbs. moving with a velocity of 100 f.-s. meets a mass of 50 lbs. moving with a velocity of 40 f.-s.; find the velocities after impact, if $e=\frac{1}{2}$.

Let v = vel. of the 20 lbs.; v_1 = vel. of the 50 lbs.

$$20\times 100-50\times 40=20v+50v_1 \quad . \quad . \quad . \quad . \qquad (1.)$$

$$v-v_1=-\tfrac{1}{2}\,(100+40) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.)$$

From these equations, $v=-50$; and $v_1=20$.

Therefore each *rebounds*, the first with a velocity of 50 f.-s., the second with a velocity of 20 f.-s.

Example iii.—Two bodies are moving in the same direction with velocities 7 and 5, and after impact their velocities are 5 and 6 respectively; determine the value of e.

Now, $v-v_1=-e\,(u-u_1)$; $\therefore\ 5-6=-e\,(7-5)$.

From which equation, $e=\frac{1}{2}$.

EXAMPLES—LIII.

1. If a mass of 12 lbs., moving with a velocity of 60 f.-s., overtake a mass of 20 lbs. moving with a velocity of 30 f.-s., find the velocities after impact. ($e=\frac{1}{2}$.)

2. If the two bodies meet, find the velocities after impact.

3. If a mass of 8 lbs. moving with a velocity of 25 f.-s. impinge on a mass of 12 lbs. at rest, find the velocities after impact. ($e=\cdot 6$.)

4. If two masses of 25 and 40, moving in opposite directions with velocities of 60 and 15 respectively, come into collision, find the velocities after impact, if $e=\frac{2}{3}$.

5. Two masses, m and $3m$, moving in opposite directions with equal velocities, come into collision; find the velocities after impact (1) if $e=1$, (2) if $e=\frac{2}{5}$.

6. A mass $4m$ impinges on a mass m at rest, and goes on with 3/4 its original velocity; find the coefficient of elasticity.

7. If two perfectly elastic masses, m and $3m$, meet when moving in opposite directions with velocities $3u$ and $8u$, find their velocities after impact.

8. A mass m impinges on a mass $\frac{1}{2}m$ at rest, and the latter starts off with a velocity 10; find the original velocity of m, if $e=\frac{1}{4}$.

9. A mass m impinges on a mass m_1 at rest, and m is brought to rest by the impact; find the ratio of m to m_1 if $e=\frac{1}{5}$.

10. A body of mass m impinges on m_1 at rest; m is brought to rest and m_1 moves on with a velocity equal to $\frac{1}{6}$ m's original velocity; find the ratio of m to m_1, and also the coefficient of elasticity.

DIRECT IMPACT UPON A PLANE AT REST.

275. If a body moving with a velocity u strike a plane at right angles, it recoils in the same line, because the mutual action is perpendicular to the plane.

Let v be the velocity of recoil,

Then, using the notation of Art. 272, we have,

$$v-v_1=-e\,(u-u_1);$$

$$\text{but } v_1=0, \text{ and } u_1=0;$$

$$\therefore\ v=-eu.$$

Of course the direction of v is *opposite* to that of u.

COR. I. If $e=1$, then $v=u$;

And we infer that when a perfectly elastic ball impinges on a plane at right angles it recoils with undiminished velocity.

COR. II. If $e=0$, then $v=0$;

And we infer that when a perfectly inelastic ball impinges on a plane at right angles there will be no recoil.

Example i. A ball falls from a height of 100 feet on a horizontal plane; to what height will it rebound if $e=\frac{2}{5}$?

$$v^2=u^2+2as;$$

$$\therefore\ v^2=2\times 32\times 100;$$

$$\therefore\ v=80 \text{ f.-s.}$$

Then the velocity of recoil $=\frac{2}{5}\times 80=32$ f.-s.

$$v^2=u^2+2as;$$

$$\therefore\ 0=(32)^2-2\times 32\times s;$$

$$\therefore\ s=16 \text{ feet.}$$

Example ii. A ball having fallen from a height h rebounds 9 feet; find h if $e=5$.

$$v^2=u^2+2as;$$

$$\therefore\ 0=u^2-64\times 9;$$

$$\therefore\ u=24.$$

Therefore by Art. 272, velocity on striking $=24\times 2=48$.

Then, $(48)^2=0+2\times 32\times h$;

From which Equation, $h=36$ feet.

EXAMPLES—LIV.

1. A particle falls from a height of 400 feet; find the velocity of rebound if $e = \frac{1}{8}$.

2. A particle falls from a height of 64 feet; and rebounds to a height of 48 feet; find e.

3. Two particles are let fall from the same height on a horizontal plane; the first rebounds to a height of 16 feet, the second to a height of 9 feet; if $e = \cdot 375$ in the latter case, find its value in the former.

4. A ball is projected from the floor of a room 18 feet high, it rebounds from the ceiling and then from the floor, and now just touches the ceiling again; find the velocity of projection if $e = \cdot 5$ in both cases.

5. A body is let fall from a height of 100 feet, and rises to a height of 60 feet after its first rebound. Find (1) the value of e, (2) the time the body is rising after its second rebound.

6. A billiard ball is driven from the side of a table, strikes the opposite cushion at right angles, and returns to the first side; compare the times of going and returning.

7. A ball falls from a height of 100 feet, and hops 4 times on a horizontal plane; find the height of the fourth hop if $e = \frac{1}{2}$.

8. A ball falls 625 feet and rebounds three times on a horizontal plane; find how high it rises after the third rebound if $e = \cdot 4$.

9. If two bodies are dropped from equal heights on a fixed horizontal plane, show that their coefficients of elasticity are as the square roots of the heights to which they rebound.

10. A particle is dropped from a height x, and rebounds again and again from a horizontal plane. If the coefficient of elasticity be $\frac{1}{2\sqrt{2}}$, show that the whole space described $= \frac{9x}{7}$.

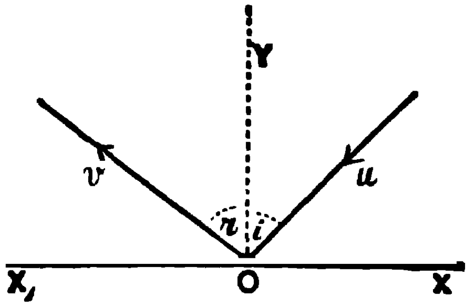
11. Generally, if e be the coefficient of elasticity, find the whole space described when a particle is let fall from a height h.

12. A mass of 40 lbs. is thrown vertically upwards with a velocity of 160 f.-s. At the height of 300 feet it comes into collision with an equal mass which has fallen through 100 feet; find the time after the impact in which each body reaches the ground, if $e = 1$.

13. A ball is thrown vertically upwards with a velocity of 20 f.-s. An equal ball is let fall at the same instant to meet it. They come into collision in five seconds. Find their original distance apart, and the velocity of each body after impact, if $e = 1$.

OBLIQUE IMPACT UPON A PLANE AT REST.

276. Let a body moving with a velocity u impinge obliquely on a fixed plane XX_1, and let its path before impact make an angle i with the normal OY, and after impact let its direction make an angle r with OY.



The angle i may be called the *angle of impact*; and the angle r the *angle of rebound.* These angles are also known as the angles of *incidence* and of *reflexion.*

Resolving the velocities along the plane and at right angles to it, we have—

Components along the plane, $u \sin i$; $v \sin r$.
" at right angles, $u \cos i$; $v \cos r$.

Now if the plane be *smooth*, there is no force acting parallel to the plane, and therefore the component of the velocity *parallel* to the plane is not changed by the impact;

$$\therefore v \sin r = u \sin i \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

And by Art. 272, $v \cos r = e . u \cos i \quad . \quad . \quad . \quad . \quad (2.)$

Dividing (1) by (2), we have—

$$\tan r = \frac{\tan i}{e};$$

$$\therefore \cot r = e \cot i \quad . \quad . \quad . \quad . \quad (3.)$$

This equation gives the *direction of the velocity* after impact.

To find the *magnitude of the velocity* after impact.

By equations (1) and (2) we have its two components, which are equal to $u \sin i$ and $eu \cos i$ respectively, and these are at right angles to each other;

$$\therefore v^2 = u^2 \sin^2 i + e^2 u^2 \cos^2 i;$$

$$\therefore v = u \sqrt{\sin^2 i + e^2 \cos^2 i} \quad . \quad . \quad . \quad . \quad (4.)$$

NOTE.—The same result might be obtained by squaring, and then adding (1) and (2).

COR. I. If $e=1$;

$$\text{Then by (3) } \cot r=\cot i.$$

And we infer that when a perfectly elastic ball impinges on a plane surface, its Angle of Impact is equal to the Angle of Rebound.

$$\text{Also by (4), } v=u\sqrt{\sin^2 i+\cos^2 i}=u.$$

And we infer that its velocity is not altered in magnitude by the impact.

COR. II. If $e=0$;

$$\text{Then by (3), } \cot r=0\,;$$
$$\therefore\ r=90^\circ$$

And by (4), $v=u\sin i$. Compare equation (1).

We therefore infer that a perfectly inelastic ball will, after impact, *run along the plane*, with a velocity equal to the component of the original velocity measured parallel to the plane.

Example i. At what angle must a ball strike a horizontal plane so that after impact its direction may be at right angles to its former path? ($e=\frac{1}{3}$.)

$$\text{By Equation (3), } \cot r=e.\cot i,$$
$$\text{but, } r=90^\circ-i\,;$$
$$\therefore\ \tan i=e.\cot i\,;$$
$$\therefore\ \tan^2 i=\tfrac{1}{3}\,;$$
$$\therefore\ \tan i=\frac{1}{\sqrt{3}}\,;$$
$$\therefore\ i=30^\circ.$$

Example ii.—A ball falling vertically strikes a plane whose inclination to the horizon is $\cot^{-1} 2$, and rebounds horizontally. Find the value of e.

Let a=angle of inclination. Then the angle of incidence (i) may be shown to be equal to a; and $r=90^\circ-i$.

$$\therefore\ \text{by Equation (3) } \cot r=e\cot i\,;$$
$$\therefore\ \tan i=e.\cot i\,;$$
$$\therefore\ 1=e.\cot^2 i\,;$$
$$\text{but, } \cot i=2, \text{ by data}\,;$$
$$\therefore\ e=\tfrac{1}{4}.$$

Example iii.—If a ball whose velocity is 60 f.-s. impinge on a fixed plane at an angle of 30° with the normal, find the direction and magnitude of the velocity after impact, if $e=\frac{1}{3}$.

Let v = velocity after impact, and θ the angle of rebound.

Then, equating horizontal components, we have—

$$60 \sin 30^\circ = v \sin\theta\,;$$

and equating vertical components, we have—

$$\tfrac{1}{3}\times 60 \cos 30^\circ = v \cos\theta\,;$$

$$\therefore\ \tan 30^\circ = \tfrac{1}{3}\tan\theta\,;$$

$$\therefore\ \tan\theta = 3\tan 30^\circ = 3\times\frac{1}{\sqrt{3}} = \sqrt{3}.$$

$$\therefore\ \theta = 60^\circ.$$

Then substitute this value of θ in either of the equations.

We select the first—

$$\therefore\ 60\sin 30^\circ = v \sin 60^\circ\,;$$

$$\therefore\ v = \frac{60\times\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{60}{\sqrt{3}} = 20\sqrt{3} = 34{\cdot}6 \text{ f.-s nearly.}$$

Example iv.—A body impinges directly on another; find the conditions that they may exchange their velocities.

By Article 274, $mu + m_1u_1 = mv + m_1v_1$ . . . . (1.)

and, $v - v_1 = -e\,(u - u_1)$ . . . . (2.)

We must have $v = u_1$, and $v_1 = u$.

Then (2) becomes $u_1 - u = -e\,(u - u_1)$;

$$\therefore\ e = 1\,;$$

and (1) becomes $mu + m_1u_1 = mu_1 + m_1u$;

or, $m\,(u - u_1) - m_1\,(u - u_1) = 0$;

or, $(m - m_1)\,(u - u_1) = 0$;

Now $(u - u_1)$ cannot be zero,

$$\therefore\ m - m_1 = 0\,;$$

or, $m = m_1$.

Therefore the conditions are, (1) the balls must have equal masses; (2) they must be perfectly elastic. Compare Art. 274.

EXAMPLES—LV.

1. A body impinges obliquely on a fixed smooth plane; determine the velocity after impact when the body is imperfectly elastic.

2. A body impinges directly on another; required their velocities after impact when the bodies are imperfectly elastic.

3. Show that, if two perfectly elastic balls of equal mass moving in the same straight line impinge upon one another, they will exchange their velocities.

4. A ball moving with velocity u impinges on a plane, and rebounds with a velocity $\frac{u}{\sqrt{2}}$; find the angles of impact and rebound, if $e=\frac{1}{\sqrt{3}}$

5. A spherical body A impinges directly upon another sphere B at rest. A remains at rest while B moves on with one-fourth of the original velocity of A. Find the value of e, and the ratio of the masses.

6. What do the results of Example 2 become if $e=0$?

7. A ball is dropped from a height of 10 feet on a plane inclined at an angle of 30°; if $e=\frac{1}{2}$, find the velocity of the rebound.

8. A ball M overtakes a ball N. They are inelastic. Their velocities are $5V$ and $4V$, and their masses as $3:2$. After impact find the velocity lost by M and gained by N.

9. A ball falls through 100 feet on a horizontal plane. If $e=\frac{1}{2}$, find its velocity after rising 10 feet, and the time of ascending 6 feet.

10. If an imperfectly elastic ball, moving on a smooth horizontal plane, after impinging on a fixed vertical hard plane, at an angle of incidence $\frac{1}{6}\pi$, bound off at an angle of reflection $\frac{1}{4}\pi$, find the value of e, and the change in its velocity.

11. If two bodies, whose masses are as $2:1$, and velocities as $1:2$ in opposite directions, undergo impact, show that if $e=\frac{5}{8}$, each ball, after impact, will move back with $\frac{5}{8}$ of its former velocity.

12. From a point at a height h, above a smooth horizontal plane, two perfectly elastic equal particles are projected at the same instant, with the same velocity v, one vertically upwards, the other vertically downwards; find the condition that they may meet again at the same point, and determine their subsequent motion.

13. One ball impinges on another ball at rest; find the condition, in order that, after impact, their directions of motion may be at right angles, e being the coefficient of elasticity.

14. A ball impinges on a smooth fixed plane, with a velocity given in magnitude and direction. If the coefficient of elasticity be e, find the velocity of the ball after impact in magnitude and direction.

15. In Example 14 determine the value of e, so that the angle of rebound may be double the angle of impact.

16. A ball A impinges directly upon an equal ball B, which strikes a cushion at right angles to the direction of motion, and, rebounding, meets A at a point half-way between the cushion and its own initial position. If the coefficient of elasticity between the balls be e, show that that between the ball and the cushion is $\frac{1-e}{3e-1}$.

17. A ball whose mass is m impinges directly on a ball whose mass is m_1 at rest, and communicates to it six times as much velocity as it retains; if $e=1$, find the ratio of m to m_1.

18. M_1, M_2, M_3, are three perfectly elastic balls placed in order in a straight line on a smooth horizontal table; M_2 impinges on M_1 at rest, and rebounding, impinges on M_3 also at rest. If M_1, M_3 move with equal velocities, prove that $M_1 - M_3 = 2M_2$.

19. Two balls, whose masses are as $2:1$ and moving in opposite directions, collide. If the first ball be brought to rest, and $e=\frac{3}{4}$, prove that their original velocities were as $7:5$.

20. If the balls in Ex. 19 were moving with velocities v and $2v$ respectively, find the velocities after impact if $e=\frac{5}{6}$.

21. The masses of 4 balls are in G. P., the common ratio being 3. The first impinges on the second at rest, the second impinges on the third at rest, and so on. Compare the velocity of the first ball before impact with the velocity communicated to the fourth ball. ($e=\frac{2}{3}$.)

22. A and B, perfectly elastic balls, moving in opposite directions, impinge directly on each other; if the mass of A be double that of B, and the velocity of B three times that of A, find the velocity of each after impact.

23. A body is formed of material such that when it strikes a plane at right angles, it will rebound with half that velocity. Having been projected vertically with a velocity v, how high will it rise after the first rebound? And supposing it to rebound again and again, find the sum of *all* the spaces it describes.

24. Using the notation of Article 274, prove that $\frac{1}{2}mu^2+\frac{1}{2}m_1u_1^2$ is greater than $\frac{1}{2}mv^2+\frac{1}{2}m_1v_1^2$.

What important conclusion may be drawn from this physical fact?

25. From the two formulæ of Art. 274, if $u_1=0$, what conclusions may be drawn about the motions of the bodies after collision?

26. A mass of 400 lbs. strikes a surface at right angles, with a velocity at 90 f.-s., and rebounds with a velocity of 30 f.-s.; find the value of the impulse.

27. A mass of 600 lbs. falls through 16 ft. and strikes the ground; find the magnitude of the blow, if there be no rebound.

28. A mass of 12 tons falls 16 feet and compresses a mass of iron 1 inch; what steady pressure would produce this effect?

29. A mass of a ton falls 10 feet and penetrates 4 inches into a sand-bank; find the average resistance offered by the sand.

30. A mass of 16 lbs. strikes a hard surface with a velocity of 120 f.-s., and rebounds with a velocity of 20 f.-s.; what blow does it give the surface?

31. A mass of 24 lbs. strikes a hard surface with a velocity of 560 f.-s., and rebounds with a velocity of 70 f.-s.; if the bodies be in contact for $\frac{1}{32}$ second, find the average stress between them.

32. A mass of 18 lbs. falls 64 feet, and rebounds from a hard horizontal surface to a height of 16 feet; find the blow delivered.

33. A mass of 90 lbs. falls 100 feet, and is brought to rest in $\frac{3}{16}$ second; find the average stress between the bodies.

34. A mass of 3 lbs. falls 4 feet, and drives a small tack $\frac{3}{4}$ inch; what mass placed on the tack would do this?

35. A mass of half a ton falls 18 feet and compresses a body 8 inches; if there be no recoil, find the mean pressure exerted.

36. A steam hammer whose mass is 18 tons falls 9 feet, and remains in contact with a mass of iron for $\frac{1}{24}$ second; find (1) the average pressure exerted; (2) the compression effected.

37. A projectile whose mass is 960 lbs. strikes a plate with a velocity of 1400 f.-s. and penetrates it 9 inches; find the average resistance afforded by the plate.

38. A mass of 10 lbs. moving with a velocity of 50 f.-s. strikes a mass of 20 lbs. moving in the same direction with a velocity of 30 f.-s.; if the coefficient of elasticity be $\frac{1}{4}$, find how much kinetic energy has been transformed into heat.

39. If in Ex. 38 the masses were 14 lbs. and 8 lbs. respectively, and they were moving in opposite directions before impact, how much kinetic energy has been transformed into heat?

CHAPTER XIV.

PROJECTILES.

277. A BODY projected with any velocity in any direction, and acted on by gravity only, is called a **Projectile.**

We are to regard the Projectile in the following investigations and Exercises as a heavy particle, and no account will be taken of the resistance caused by the air.

It may be remarked that the resistance offered by the air to the motion of a Projectile is so great that its neglect in the following investigations renders the results arrived at of little practical value. The study of the theory, however, affords a valuable exercise in the principles of the preceding chapters.

The subject is treated in this Book without introducing Conic Sections.

278. We begin by defining certain terms which are constantly used in connection with this subject.

DEF.—The **Angle of Elevation** is the inclination of the direction of projection to the horizontal plane through the point of projection.

We denote it by α. (*See* Fig. of Art. 281.)

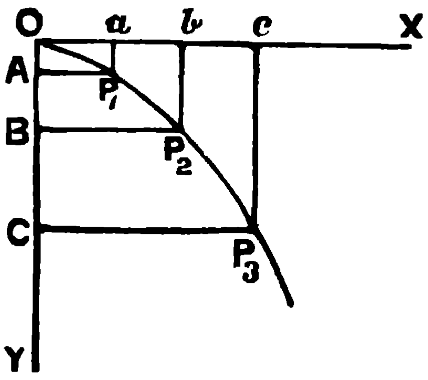
DEF.—The **Range** of a Projectile is the distance between the point of projection and the point where the projectile strikes any particular plane through the point of projection.

OR is the range. (*See* Fig. of Art. 281.)

DEF.—The **Time of Flight** of a Projectile is the interval between its projection and its striking the plane. It is usually denoted by T.

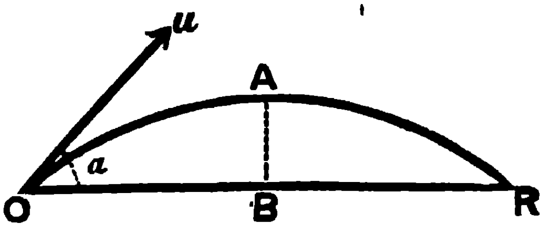
279. If a body has two uniform velocities, its path will be a straight line (Art. 136). If a body be thrown vertically upwards or downwards, although it is subject to the action of gravity, yet its path is also a straight line, there being no force to take it out of the vertical line; but if a body be projected with uniform velocity in any other direction and gravity act on it, it will describe a *curved* path.

If a body be projected horizontally along OX, with a uniform velocity which will cause it to describe the *equal* spaces Oa, ab, bc, ... in successive seconds, then if gravity act on it, the body will fall in successive seconds through the spaces OA, AB, BC, ... which by Article 69, are proportional to the numbers 1, 3, 5, ... Therefore at the end of 1, 2, 3, ... seconds the body will be found at . . . . P_1, P_2, P_3 . . . . If we take smaller intervals of time we can determine intermediate positions of the body, and if the number of such intervals be infinite, the path will be a continuous curve, and this path is known to be a *Parabola*.



280. In considering motion in a curved line situated in a vertical plane, the point for the student to notice particularly is that the velocity at any instant may be supposed to be the resultant of two velocities, one *horizontal*, the other *vertical*, and that each of these two motions may be treated quite independently of the other. The horizontal motion does not interfere with the action of gravity (*see* Art. 105), and the action of gravity cannot affect motion taking place at right angles to its own direction.

281. Let a particle be projected with a velocity $= u$ in a direction α with the horizon.



Resolving vertically and horizontally, we obtain—

The Horizontal Component $= u \cos \alpha$.
The Vertical Component $= u \sin \alpha$.
The Horizontal Acceleration $= 0$.
The Vertical Acceleration $= -g$.
The Horizontal Distance in t secs. $= u \cos \alpha \times t$.
The Vertical ,, ,, $= u \sin \alpha \times t - \frac{1}{2}gt^2$.
The Horizontal Velocity after t secs. $= u \cos \alpha$.
The Vertical ,, ,, $= u \sin \alpha - gt$.

282. *To find the Velocity and Direction of its path after t seconds.*

Velocity.—The Hor. Vel. after t seconds $= u \cos \alpha$.
The Vert. ,, ,, $= u \sin \alpha - gt$.

Let v be the velocity required.

Then, these two components being at right angles,

$$v = \sqrt{u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2}.$$

Direction.—Let θ be the inclination of the velocity to the horizon at the instant.

$$\text{Then, } \tan \theta = \frac{\text{vertical velocity}}{\text{horizontal velocity}},$$

$$\therefore \tan \theta = \frac{u \sin \alpha - gt}{u \cos \alpha}.$$

283. *To find when the projectile is at its Highest Point (A).*

This happens when it has no vertical velocity.

We use the formula, $v=u+at$;

$$\therefore 0=u\sin\alpha-gt;$$

$$\therefore t=\frac{u\sin\alpha}{g}.$$

284. *To find the Greatest Height attained (AB).*

This happens when it has lost all its vertical velocity.

We use the formula, $v^2=u^2+2as$;

$$\therefore 0=(u\sin\alpha)^2-2gs;$$

$$\therefore s=\frac{u^2\sin^2\alpha}{2g}.$$

285. *To find the Time of Flight on a horizontal plane.*

The projectile returns to the horizontal plane through the point of projection in double the time taken to reach its highest point. (*See* Articles 71 and 105.)

$$\text{Time to highest point}=\frac{u\sin\alpha}{g}\text{ (by Art. 283)};$$

$$\therefore T=\frac{2u\sin\alpha}{g}.$$

Or thus: after the lapse of the Time of Flight the projectile strikes the plane, and therefore its height $=0$.

Using $s=ut+\frac{1}{2}at^2$, we have

$$0=u\sin\alpha \,.\, T-\tfrac{1}{2}gT^2;$$

$$\therefore T=\frac{2u\sin\alpha}{g}.$$

286. *To find the Range on a horizontal plane.*

This depends on the Uniform Horizontal Velocity and the Time of Flight.

$$\therefore \text{Range}=OR=u\cos\alpha\times T;$$

$$\text{But, } T=\frac{2u\sin\alpha}{g}\text{ (Art. 285)};$$

$$\therefore \text{Range}=u\cos\alpha\times\frac{2u\sin\alpha}{g}=\frac{u^2}{g}\sin 2\alpha.$$

287. *To find the Velocity at the Highest Point.*

The Vertical velocity $=0$.

There remains therefore only its horizontal velocity;

$\therefore$ Velocity at its highest point $=u\cos\alpha$;

and at that instant the Projectile is moving *horizontally*.

288. *To find the Velocity at the end of the Horizontal Range.*

The vertical velocity on reaching the plane is found by the formula $v=u+at$.

$$\therefore \text{ Vert. Vel.} = u\sin\alpha - gT$$
$$= u\sin\alpha - g\frac{2u\sin\alpha}{g} \text{ (Art. 285)}$$
$$= -u\sin\alpha.$$

The Horizontal Velocity is the same at both ends: viz. $u\cos\alpha$.

Therefore the *magnitude* of the velocity is the same at both ends of the Range; and its direction at the end of the Range is as much *depressed* below the horizontal plane as it was *elevated* above it at the point of projection; because, if θ and θ_1 be the directions at the extremities of the Range, it is evident that—

$$\tan\theta = \frac{u\sin\alpha}{u\cos\alpha}, \text{ and } \tan\theta_1 = \frac{u\sin\alpha}{u\cos\alpha}.$$

289. *To find the Velocity after describing a given vertical height.*

Let the given height $=h$.

$$\text{Vertical Velocity} = \sqrt{u^2\sin^2\alpha - 2gh},$$
$$\text{Horizontal ,, } = u\cos\alpha;$$
$$\therefore \text{ Velocity required} = \sqrt{u^2\sin^2\alpha - 2gh + u^2\cos^2\alpha}$$
$$= \sqrt{u^2 - 2gh}.$$

290. *To find the Direction of the motion at a given height.*

Let the given height $=h$.

Let the direction of motion make an angle θ with the horizon.

$$\tan\theta = \frac{\text{vertical velocity}}{\text{horizontal velocity}} = \frac{\sqrt{u^2\sin^2\alpha - 2gh}}{u\cos\alpha}.$$

291. *To find the Angle of Elevation which with a given velocity of projection will give the greatest Horizontal Range.*

$$\text{The Range}=\frac{u^2}{g}\sin 2a. \quad \text{(Art. 286.)}$$

Because u and g are constant, the Range will be greatest

when $\sin 2a$ is greatest;

i.e. when, $\sin 2a=1$;

or, when $2a=90^\circ$;

or, when $a=45^\circ$.

292. *To find the Maximum Range.*

$$\text{The Range}=\frac{u^2}{g}\sin 2a. \quad \text{(Art. 286.)}$$

The Range is a maximum when $a=45^\circ$. (Art. 292.)

Let $a=45^\circ$;

$$\therefore \text{Range}=\frac{u^2}{g}.$$

Example i.—A particle is projected at an angle of elevation of 30° with a velocity of 240 f.-s.; find its height after three seconds.

The initial vertical component $=240\sin 30^\circ=120$ f.-s.

$$s=ut+\tfrac{1}{2}at^2;$$

$$\therefore s=120\times 3-16\times 9;$$

$$\therefore \text{Height}=216 \text{ feet.}$$

Example ii.—A particle is projected at an angle of elevation of 60° with a velocity of 1000 f.-s.; find (1) the Time of Flight, (2) the Range.

Time of Flight. Vert. velocity $=1000\sin 60^\circ=500\sqrt{3}$ f.-s.

$$v=u+at;$$

$$\therefore 0=500\sqrt{3}-32t;$$

$$\therefore t=\frac{500\sqrt{3}}{32};$$

$$\therefore T=\frac{1000\sqrt{3}}{32}=54 \text{ seconds nearly};$$

Range. Hor. Vel. $=1000\cos 60^\circ=500$ f.-s.

Range $=500\times 54=27{,}000$ feet.

Example iii.—The Range of a projectile is 1600 feet, the angle of elevation 30°; find its velocity of projection.

Let u = Vel. of projection.

$$\text{Vert. velocity} = u \sin 30^\circ = \frac{u}{2}.$$

$$\text{Hor.} \quad " \quad = u \cos 30^\circ = \frac{u\sqrt{3}}{2}.$$

$$\text{Time of Flight} = \frac{\text{Hor. Range}}{\text{Hor. Vel.}} = \frac{1600 \times 2}{u\sqrt{3}}.$$

$$\therefore \text{Time to highest point} = \frac{1600}{u\sqrt{3}};$$

$$\text{Now, } v = u + at;$$

$$\therefore 0 = \frac{u}{2} - \frac{32 \times 1600}{u\sqrt{3}};$$

$$\therefore u^2 = \frac{64 \times 1600}{\sqrt{3}}.$$

From which equation, $u = 243$ f.-s., nearly.

Example iv.—The horizontal Range is 1000 feet, the Time of Flight is 15 seconds; find the angle of elevation and the greatest height attained.

The body falls in 7½ seconds from its highest point.

$$s = ut + \tfrac{1}{2}at^2 = 0 + \frac{16 \times 225}{4} = 900;$$

$\therefore$ *Greatest height* = 900 feet.

$$\text{Vertical velocity, } v = u + at,$$

$$v = 0 + \frac{32 \times 15}{2} = 240 \text{ f.-s.}$$

$$\text{Horizontal velocity} = \frac{\text{Hor. Range}}{\text{Time of Flight}} = \frac{1000}{15} = \frac{200}{3} \text{ f.-s.}$$

Let a = angle of elevation.

$$\therefore \tan a = \frac{240}{\frac{200}{3}} = 3\cdot6;$$

$$\therefore a = \tan^{-1}(3\cdot6).$$

Example v.—A body is projected with a velocity of 256 f.-s. at an angle of elevation = a, and passes horizontally over a wall 64 feet high; find the value of a.

The vertical velocity = 256 sin a.

At the height 64 feet it has lost all vertical velocity.

$$v^2 = u^2 + 2as,$$

$$\therefore 0 = (256 \sin a)^2 - 64 \times 64;$$

$$\therefore \sin^2 a = \frac{64 \times 64}{(256)^2} = \frac{1}{16};$$

$$\therefore \sin a = \tfrac{1}{4};$$

$$\therefore a = \sin^{-1}(\tfrac{1}{4}).$$

Example vi.—A body is projected with a velocity of 500 f.-s.; find its greatest range on a horizontal plane.

By Art. 291, the angle of elevation must $=45^\circ$.

$$\therefore \text{Vert. velocity} = \text{Hor. vel.} = \frac{500}{\sqrt{2}} = 250\sqrt{2} \text{ f.-s.}$$

Time to highest point, $v=u+at$;

$$\therefore 0 = 250\sqrt{2} - 32t;$$

$$\therefore t = \frac{250\sqrt{2}}{32};$$

$$\therefore \text{Time of Flight} = \frac{500\sqrt{2}}{32} \text{ secs.};$$

$$\therefore \text{Range} = \text{Hor. vel.} \times \text{Time of Flight} = \frac{500\sqrt{2}}{32} \times 250\sqrt{2}$$

$$= 7812{\cdot}5 \text{ feet.}$$

EXAMPLES—LVI.

1. A body is projected with a velocity of 960 f.-s. at an angle of elevation $=45^\circ$; find its horizontal Range.

2. A body is projected at an angle of elevation $=30^\circ$, with a velocity of 1200 f.-s.; find (1) the greatest height attained, (2) the Time of Flight, (3) the horizontal Range.

3. A body is projected at an angle of elevation $=45^\circ$ with a velocity of 1600 f.-s.; find (1) the greatest height attained, (2) the Time of Flight, (3) the horizontal Range.

4. In Example 2, find the direction of the motion after three seconds.

5. In Example 3, find the vertical velocity after four seconds.

6. A body is projected at an angle of elevation $\sin^{-1}(\frac{1}{8})$ with a velocity of 600 f.-s., find the horizontal Range.

7. A body is projected at an angle of elevation $=60^\circ$, and in six seconds passes horizontally over an object; find the velocity of projection.

8. A body is projected with a velocity of 320 f.-s., and has a horizontal Range $=3200$ feet; find the angle of elevation.

9. A cricketer throws a ball with a velocity of 75 f.-s. at an angle of elevation $=30^\circ$; how far does he throw it?

10. At what angle must he throw it to make the longest possible 'throw'?

11. If he throw it at this angle, find the velocity if he desires it to reach the wicket-keeper, who is distant 80 yards.

12. In the experimental proof of the 110-ton gun the muzzle velocity was 2140 f.-s.; find the horizontal Range if the gun were elevated at an angle of 45°.

13. If the horizontal Range be equal to the greatest height attained, find the angle of elevation.

14. If the horizontal Range be equal to the height due to the velocity of projection, find the angle of elevation.

15. The greatest horizontal Range is 200 feet; find the velocity of projection.

16. The greatest horizontal Range is 4050 feet; find the velocity of projection.

17. If two bodies, projected at the same instant from different points in the same horizontal plane, be at any moment at the same height above the plane, show that their heights are the same at any subsequent time.

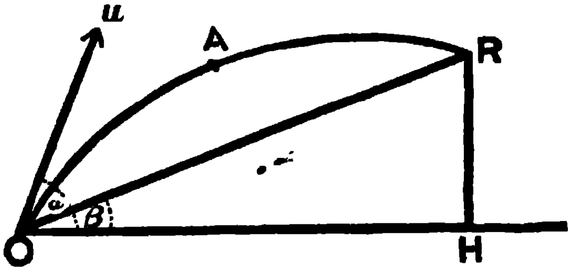
18. If the horizontal Range be 3 times the greatest height attained, find the angle of elevation.

19. A ball is projected at an angle of elevation of 30° with a velocity due to a fall of 300 feet; find the greatest height reached by the ball.

20. A body is projected at an angle of elevation $=\alpha$, with a velocity of 32 f.-s.; in what time will the direction of its motion be $\frac{1}{2}\alpha$?

293. We have considered the Range, Time of Flight, etc., on a horizontal plane. We have next to deal with the same class of problems when the plane through the point of projection is *inclined at an angle with the horizon.*

294. When the direction of projection makes an angle with an inclined plane through the point of projection; to find (i.) the *Time of Flight;* (ii.) *the Range on the plane;* (iii.) the *Greatest Distance of the Projectile from the plane.*



The angle of elevation $=\alpha$;
The inclination of the plane $=\beta$.

Resolve u along the plane OR and at right angles to it;
,, g ,, ,, ,, ,,
Component of u along $OR=u\cos(\alpha-\beta)$;
,, g ,, $=g\sin\beta$;
,, u at right angles to $OR=u\sin(\alpha-\beta)$;
,, g ,, ,, $=g\cos\beta$.

The motion along OR is independent of the motion at right angles to it. (*See* Art. 280.)

(i.) *Time of Flight.*

When the Projectile is at A, the greatest distance from CR, its velocity at right angles to $OR=0$.

$$v=u+at;$$

$$\therefore\ 0=u\sin(\alpha-\beta)-g\cos\beta\cdot t;$$

$$\therefore\ t=\frac{u\sin(\alpha-\beta)}{g\cos\beta};$$

$$\text{or, } T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

Or, the method at the end of Article 285 may be used.

(ii.) *Range.*

The Range $=OR$;

and, $OR=OH\sec\beta$;

now, $OH=T\times u\cos\alpha$;

$\therefore\ OR=T\times u\cos\alpha\sec\beta$,

$$\text{but, } T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta};$$

$$\therefore\ OR=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

(iii.) *The greatest distance above the plane.*

Let this distance $=x$.

$$v^2=u^2+2as;$$

$$\therefore\ 0=u^2\sin^2(\alpha-\beta)-2g\cos\beta\cdot x;$$

$$\therefore\ x=\frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta}.$$

295. *If the velocity of projection be given, to find for what Angle of elevation the Range is greatest on an inclined plane passing through the point of projection.*

By Art. 294 (ii.) the Range$=\dfrac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$.

For what value of α is this greatest?

$$\text{Now, } 2\cos\alpha\sin(\alpha-\beta)=\sin(2\alpha-\beta)-\sin\beta;$$

$$\therefore \cos\alpha\sin(\alpha-\beta)=\tfrac{1}{2}\{\sin(2\alpha-\beta)-\sin\beta\}.$$

This is greatest when $\sin(2\alpha-\beta)$ is greatest;

i.e. when $\sin(2\alpha-\beta)=1$;

or, when $2\alpha-\beta=90°$;

or, when $\alpha=\dfrac{90°+\beta}{2}$.

Therefore when the angle of elevation$=\frac{1}{2}(90°+\beta)$, the Range is greatest on the plane *OR*.

NOTE.—The direction of the velocity of projection therefore bisects the angle between the horizontal line and that drawn at right angles to the plane.

296. *To find the condition that a Projectile may strike at right angles an inclined plane through the point of projection.*

This happens if it has *no velocity parallel to the direction* of the plane, at the instant of striking.

Its velocity due to u along $OR=u\cos(\alpha-\beta)$;

" g " $=-g\sin\beta\times t$;

now, $v=u+at$;

here t the Time of Flight$=\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$; Art. 294 (i.)

$$\therefore 0=u\cos(\alpha-\beta)-g\sin\beta\times\frac{2u\sin(\alpha-\beta)}{g\cos\beta};$$

$$\therefore \frac{2\sin\beta\sin(\alpha-\beta)}{\cos\beta}=\cos(\alpha-\beta);$$

$$\therefore 2\tan(\alpha-\beta)=\cot\beta.$$

And this is the condition required.

297. *To find the Condition that a Projectile may strike an inclined plane horizontally.*

This will happen when it has no *vertical* velocity at the instant of striking.

The initial vertical velocity $= u \sin \alpha$.

This must be destroyed in the Time of Flight,

$$\text{Now, } v = u + at;$$

$$\therefore 0 = u \sin \alpha - \frac{g.2u \sin(\alpha - \beta)}{g \cos \beta};$$

$$\therefore \sin \alpha \cos \beta = 2 (\sin \alpha \cos \beta - \cos \alpha \sin \beta);$$

$$\therefore 2 \cos \alpha \sin \beta = \sin \alpha \cos \beta;$$

$$\therefore 2 \cot \alpha = \cot \beta.$$

And this is the condition required.

Example i.—A particle is projected with a velocity of 800 f.-s. at an angle of elevation $=60°$; find its Range up a plane whose inclination is 30°.

Comp. of Velocity at right angles to OR (*see* Fig. of Art. 294)

$$= 800 \sin 30° = 400 \text{ f.-s.}$$

Comp. of Gravity, at right angles to $OR = 32 \cos 30° = 16\sqrt{3}$.

To find time to greatest distance from OR

$$v = u + at;$$

$$\therefore 0 = 400 - 16\sqrt{3}.t;$$

$$\therefore t = \frac{25}{\sqrt{3}}$$

$$\therefore T = 2t = \frac{50}{\sqrt{3}}.$$

Now, $OH = 800 \cos 60° \times T = 800 \times \frac{1}{2} \times \frac{50}{\sqrt{3}} = \frac{20{,}000}{\sqrt{3}}$.

Then, $OR = OH \sec 30° = \frac{20{,}000}{\sqrt{3}} \times \frac{2}{\sqrt{3}} = 13{,}333\frac{1}{3}$ feet.

Example ii.—A body is projected in a direction making an angle of 30° with a plane whose inclination to the horizon is 45°, the Range being 250 feet; find (1) the velocity of projection, and (2) the Time of Flight.

To find Velocity.—Let $u =$ Velocity of projection.

Velocity at right angles to $OR = u \sin 30° = \frac{u}{2}$.

Comp. of Gravity at right angles to $OR = 16\sqrt{2}$.

Now, $v = u + at$;

$$\therefore 0 = \frac{u}{2} - 16\sqrt{2}.t;$$

$$\therefore t = \frac{u}{32\sqrt{2}};$$

$$\therefore T = 2t = \frac{u}{16\sqrt{2}}.$$

$OH = 250 \cos 45°$; also, $OH = u \cos 75° \times T$;

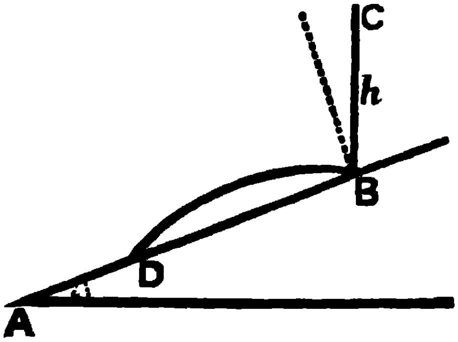
$$\therefore 250 \times \frac{1}{\sqrt{2}} = u.\frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{u}{16\sqrt{2}}.$$

From which, $u = 124$ ft., nearly.

To find Time of Flight.—Then $T = \frac{u}{16\sqrt{2}} = \frac{124}{16\sqrt{2}}$.

From which, $T = 5{\cdot}6$ sec., nearly.

Example iii.—A ball falls from a height h on a plane whose inclination is i; find the range on the plane after the first rebound, when the coefficient of elasticity is e.



Velocity on striking at $B = \sqrt{2gh}$.

Let this be denoted by u.

Consider motion parallel to the plane.

The velocity along the plane is not changed, and $= u \sin i$.

Consider the motion at right angles to the plane.

Vel. of approach $= u \cos i$,

,, recoil $= e.u \cos i$.

To find time of flight, $v = u + at$;

$$\therefore 0 = eu \cos i - g \cos i.t;$$

$$\therefore t = \frac{eu}{g};$$

$$\therefore T = \frac{2eu}{g}.$$

The range BD is given by the formula, $s = ut + \frac{1}{2}at^2$.

$$\therefore R = u \sin i \times T + \tfrac{1}{2}g \sin i \times T^2,$$

$$= u \sin i \times \frac{2eu}{g} + \tfrac{1}{2}g \sin i \times \frac{4e^2u^2}{g^2};$$

and this reduces to $4eh\,(1+e) \sin i$.

EXAMPLES—LVII.

1. A body is projected at an angle of elevation $=75°$ and with a velocity of 96 f.-s.; find when it will be 30 feet above the point of starting, and the distance between its positions at those moments.

2. A particle is projected at an angle of elevation $=45°$ and with a velocity of 100 f.-s.; find its Range and Time of Flight on a plane through the point of projection whose inclination is 30°.

3. Find the maximum Range on a plane inclined at an angle of 30°, if the velocity of projection be 300 f.-s.

4. With what velocity must a projectile be fired at an angle of elevation $=30°$, in order that its Range may be 2600 feet on a plane whose angle of inclination $=\sin^{-1}\left(\frac{1}{30}\right)$?

5. If the greatest Range of a projectile on a horizontal plane be 5881·5 feet, find its greatest Range up a plane whose inclination is 30°.

6. A particle is projected with a velocity u at an angle of elevation α; find (1) its least velocity, (2) the direction of its path at that moment.

7. In the last Example, find the distance of the projectile from the point of projection at that instant.

8. If the velocity of projection be the same, compare the greatest Ranges on a horizontal plane, up a plane whose inclination is 30°, and up a plane whose inclination is 45°.

9. Find the velocity of projection and the angle of elevation in order that a projectile may pass through a point with no vertical velocity.

10. Compare the greatest Ranges on a horizontal plane and up an inclined plane whose inclination is 60°, the velocity of projection being the same in both cases.

11. A particle, being projected with a velocity u at an angle of elevation α, just grazes the edges of a cube of which the edge is x, and which stands on a horizontal plane; show that

$$x^2g^2=4u^2\cos^2\alpha\,(u^2\sin^2\alpha-2xg).$$

12. A stone is thrown from the top of a tower 32 feet high at an angle of elevation $=30°$, with a velocity of $\frac{32}{\sqrt{3}}$ f.-s.; find its velocity and direction of motion on striking the horizontal plane through the foot of the tower.

13. If the greatest Range straight down a plane be three times the greatest Range up, show that the inclination of the plane is 30°.

14. If a gun be fired at an elevation θ, where θ is very small, so as to hit a mark on the ground at a distance a, and b be the range when fired at an angle of 45°, prove that $\theta = \frac{a}{2b}$ nearly.

15. A particle is projected with a velocity of 40 f.-s. from a given point A in order to hit a point 10 feet in front of A, and 15 feet above A; find the angle of elevation.

16. Two particles are projected with equal velocities at the same instant from the same point. Their horizontal Ranges are the same, and their Times of Flight are as $\sqrt{3} : 1$; find the angles of elevation.

17. A particle projected at an angle of elevation $= 75°$ fell, at a distance of 250 feet, on a plane passing through the point of projection whose inclination was 45°; find the velocity of projection and the Time of Flight.

18. In what time will a shell, projected at an angle of elevation $= 30°$, obtain a Range of 3000 feet on a horizontal plane?

19. A particle is let fall on an inclined plane, and its range on the plane after rebounding is $\frac{4}{5}$ the space fallen through before impact; if $e = \frac{1}{2}$, find the inclination of the plane.

20. A particle falls from a height x on a plane whose inclination is 30°; after rebounding it strikes the plane again; show that its Time of Flight on the plane is equal to twice the time of its fall, if $e = 1$.

21. In Ex. 20 find the Range on the plane.

22. A ball is projected horizontally from a height of 100 feet with a velocity of 100 f.-s. After its first rebound it describes a horizontal distance of 40 feet; find the value of e.

23. A ball is projected from a point in a horizontal plane and rebounds twice. If the second Range is equal to the greatest height to which the ball rises, show that $\tan \alpha = 4e$, if α is the angle of projection and e is the coefficient of elasticity.

24. A heavy particle is projected from the foot of an inclined plane at an angle θ to the plane. The inclination of the plane to the horizon is α. Show that, if the particle when it strikes the plane is moving perpendicularly to it, $\tan \theta = \frac{1}{2} \cot \alpha$.

25. If a particle be projected *in vacuo* with a velocity V at an angle α to the horizon, find the greatest height to which it attains, and the time which it takes in so doing.

Apply your formulæ when $\alpha = 15°$, and $V = 500$ f.-s.

26. An inclined plane passes through the point of projection of a projectile, and is at right angles to the plane of motion; find the Time of Flight, and the Range on the plane.

27. If, in Ex. 26, i be the inclination of the plane, and if the projectile impinge perpendicularly on the plane, show that the Range on the plane is $\frac{2v^2(\sin i)}{g(1+3\sin^2 i)}$, where v is the velocity of projection.

28. A man standing a feet from a wall whose height is h, fires a rifle. The bullet just clears the wall, and reaches the ground a feet beyond it; prove that the elevation of the rifle must have been $\tan^{-1}\left(\frac{2h}{a}\right)$.

29. AB is the Range of a projectile on a horizontal plane; show that if t be the time from A to any point of P of the projectile's path, and t_1 the time from P to B, the vertical height of P above AB is $\frac{1}{2}gtt_1$.

(*Note.*—This useful result is known as the 'Student's Formula.')

30. Determine the angle of elevation so that the Range on a plane inclined at an angle of 15° may be the greatest possible.

31. If a body be projected in such a direction that the Range on a horizontal plane is the greatest, show that the direction of motion on striking the plane is perpendicular to the direction of projection.

32. If the velocity of projection be given, show that the horizontal Range is the same, whether the angle of elevation be $(\frac{1}{4}\pi+a)$, or $(\frac{1}{4}\pi-a)$. Prove this, and compare the Times of Flight.

33. Show that the greatest possible Range on a plane whose inclination is i of a body projected with a velocity due to a fall from a height h is $\frac{2h}{1+\sin i}$.

34. If R be the horizontal Range, and T the Time of Flight, show that when the angle of elevation is a, $\tan a=\frac{gT^2}{2R}$.

35. A particle is projected at an angle of elevation $=a$, with a velocity $=u$, and it strikes a wall situated at a distance x feet; at what height does it strike it?

36. Having given the Range (R) on a horizontal plane and the initial velocity (u), find the angle of elevation and the Time of Flight.

37. Find the condition that for a given horizontal range the velocity of projection may be a minimum.

38. With a given initial velocity, show that the greatest Range on a horizontal plane is half the greatest range down a plane whose inclination is 30°.

CHAPTER XV.

WORK AND ENERGY.

298. THE effect of a force considered as acting on a body for a given *time* is called *Momentum;* whereas the effect of a force considered as acting on a body through a given *space* is termed **Work**.

299. An *Agent is said to do Work* when it causes the point of application of the force which it exerts to move through a certain space. *E.g.*, a carpenter does work in planing wood; gravity does work when a stone falls; a projectile does work when it penetrates a mound of earth.

It will be noticed that both *force* and *motion* are necessary for the production of work.

Hence it follows that a stone resting on the ground does no work, because, although there is force, yet there is no motion. Steam at high pressure in a boiler is doing no work for a similar reason. A body which is moving with uniform velocity is doing no work, because, although there is motion, yet there is no force. (*See* Art. 92.)

300. Work *is measured* by the product of the numbers denoting the force and the distance through which it is exerted.

If F denote the magnitude of a force, and s denote the space through which it acts, then

No. of Units of Work done $= Fs$.

301. Now for certain purposes it is very convenient to measure force in terms of a Pound Weight as unit. Engineers in this country invariably adopt the gravitation measure of force. The **Unit of Work** used by British engineers is called a foot-pound.

Def.—*A* **Foot-pound** *is the work done in overcoming through* 1 *foot the resistance caused by gravity on a pound of matter.* In other words, it is the work done in lifting a pound of matter a foot high, or in overcoming a Pound Weight through a distance of one foot.

When several forces act on a body, some in the direction of the body's motion, and the rest in the opposite directions, the former may be said to do *positive* work, the latter *negative* work; and the *total* work done on the body is the Algebraical Sum of these works.

We might have defined a 'Foot-pound' as the work done by gravity on the body when a pound of matter falls vertically through one foot.

302. A foot-pound is a *gravitation* unit, and is a suitable measure of the work done by an engine or by a labourer where absolute accuracy is unnecessary. The objection to this unit for the scientific measurement of Work is, that its value depends on the local value of gravity. To measure Work in a uniform and consistent manner, we must use as unit the work done by the Kinetic Unit of Force exerted through the Unit of Space. (*See* Art. 310.)

For the present the student will consider Work as measured in Foot-Pounds.

Example i.—Find the work done by gravity when a mass of 6 lbs. falls through a height of 10 feet.

Units of work expended $= Fs = 6 \times 10 = 60$ ft.-lbs.

Example ii.—How many units of work are expended in raising 20 lbs. through 80 feet?

No. of units of work $= 20 \times 80 = 1600$ ft.-lbs.

Example iii.—If the pressure on a piston be 20 lbs. per sq. in., the area of the piston be 40 sq. in., and the length of the stroke be 2 feet, how many units of work are yielded per stroke?

No. of units of work $= 20 \times 40 \times 2 = 1600$ ft.-lbs.

303. Now, on comparing the results of the last two examples, we infer that any agent which raises 20 lbs. through 40 feet will do the same amount of work as is done by the engine at each stroke.

Yet if the agent in Example ii. *takes longer to do this work* than the engine in Example iii., we know that the *power* of the two is not the same.

Def.—The **Power** of an agent is the *rate* at which it can do work.

Hence to make a complete comparison between the working power of two agents, we must take account of the *time* in which the work is done.

304. Thus in the examples referred to, if the agent in the first raise 200 lbs. through 80 feet *in a minute*, and the engine make 10 strokes *a minute*, each will perform 16,000 units of work *in the same time*, and therefore the two agents are capable of yielding exactly the same amount of work.

It follows then, by our definition, that the true measure of the working power of any agent is the number of units of work yielded in a given time.

305. Hence in measuring the working power of machinery, we must consider not only the work done, but also the *time* in which the work is done. When a comparison of the working powers of machines is required, the following definition becomes necessary.

Def.—*A* **Horse-Power** *is the power of doing* 33,000 *foot-pounds per minute*; or, 550 *foot-pounds per second.*

An engine of x horse-power can therefore perform $x \times 33{,}000$ foot-pounds per minute; or, $550x$ foot-pounds per second.

Formula for working Train Examples.

H.P. × 550 = *Total Resistance* × *Speed of train per second.*

Each side of this Equation is the measure of the work done per second.

Example iv.—Find the horse-power of an engine which will raise 660 cubic feet of water in an hour from a depth of 480 feet, a cubic foot of water weighing $62\frac{1}{2}$ lbs.

Let x = horse-power required.

Work yielded by the engine in 1 hour $= x \times 33000 \times 60$ ft.-lbs.

Work to be done $= 660 \times 62\frac{1}{2} \times 480$ ft.-lbs.

These Works are equal.

$\therefore x \times 33000 \times 60 = 660 \times 62\frac{1}{2} \times 480$; $\therefore x = 10$ H.P.

Example v.—Find the H.P. of a locomotive which moves a train of 50 tons at the rate of 30 miles an hour along a level railroad, the resistance from friction and the air being 16 lbs. weight per ton.

A velocity of 30 miles an hour = 2640 feet per minute.

Let x = horse-power of the engine;

$\therefore x \times 33000 = 16 \times 50 \times 2640$; $\therefore x = 64$ H.P.

Example vi.—Find the H.P. of a locomotive which moves a train of 44 tons at the rate of 30 miles an hour up an incline of 1 in 100, the resistance caused by friction being 15 lbs. weight per ton.

The work done per minute = (weight lifted + friction overcome) × 2640.

The part of the weight lifted $= (44 \times 2240 \sin i)$ lbs. weight.

The resistance overcome $= 44 \times 15$ lbs. weight.

Let x = horse-power of the engine;

$\therefore x \times 33000 = (44 \times 2240 \times \frac{1}{100} + 44 \times 15) \times 2640.$

From which equation, $x = 131 \cdot 6$ H.P.

306. THEOREM.—*The work done in drawing a body up a smooth inclined plane is equal to the work done in raising the same body through the vertical height of the plane.*

Let W be the weight of the body, h the height, and l the length of the plane.

The resistance = component of the weight along the plane

$$= W \sin i.$$

The work done in drawing the body along the plane

$$= W \sin i \times \text{length of the plane.}$$

$$= W \times \frac{h}{l} \times l.$$

$$= Wh.$$

= weight × height of the plane.

= work done in lifting the body through the vertical height of the plane. (Q.E.D.)

307. THEOREM.—*The work done in drawing a body up a rough inclined plane is equal to the work done in drawing the body along a rough horizontal plane equal to the base of the plane, together with the work done in lifting the body through the height of the plane.*

Let W be the weight of the body,
and μ the coefficient of friction.

Then the Work done in drawing the body up the plane

$$=(W\sin i+\mu W\cos i)l$$
$$=W(h+\mu b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Where h is the height, l is the length, and b is the base of the plane.

The Work done in drawing the body along the horizontal

$$\text{plane}=\mu Wb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

The Work done in lifting the body up a distance h

$$=Wh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

The sum of the Works done in (2) and (3) is evidently equal to the Work done in (1). (Q.E.D.)

If a man weighing 14 stones ran up a staircase so as to rise vertically 3 feet per second, he would do 588 ft.-lbs. per second. If he kept this up for a minute, he would do 35280 ft.-lbs., *i.e.* he would do more work for that time than a horse can keep up, and, it is estimated, not less than 75 times the work which a labourer toiling continuously can perform when lifting earth with a spade.

EXAMPLES—LVIII.

NOTE.—A cubic foot of water = $62\frac{1}{2}$ lbs.

1. Find the work expended in raising 50 lbs. vertically through 6 feet.

2. Find the H.P. of an engine which will raise 352 cubic feet of water per minute from a depth of 75 feet.

3. Find the H.P. of an engine which will raise 6000 cubic feet of water per hour from a depth of 500 feet.

4. A locomotive draws a train of 70 tons at the rate of 15 miles an hour; the resistances amount to 8 lbs. weight per ton; find the H.P. of the engine.

5. An engine is required to raise 100, 80, and 30 cubic feet of water per minute from depths of 180, 150, and 110 feet respectively; find the H.P. of the engine.

6. Find the speed at which a locomotive of 35 H.P. will draw a train of 80 tons along a level railroad, the resistances amounting to 10 lbs. weight per ton.

7. An engine whose H.P. is 30 will draw a train at the rate of 30 miles an hour against a resistance amounting to 9 lbs. weight per ton; find the mass of the train and engine.

8. An engine of 25 H.P. draws a train of 45 tons at the rate of 20 miles an hour; find the resistance per ton.

9. If an engine draw a train of 50 tons at the rate of 50 miles an hour against a resistance of 8 lbs. weight per ton, find its horse-power?

10. An engine of 300 H.P. draws a train of 112 tons up a slope rising 1 in 280, the friction amounting to 12 lbs. weight per ton; at what speed is the train travelling?

11. An engine of 267 H.P. draws a train of 89 tons at a uniform speed of $28\frac{1}{13}$ miles per hour up a slope; find the slope if the resistance due to friction amount to the weight of 11 lbs. per ton.

308. THEOREM.—*If a number of weights be raised through different heights, the amount of work done is equal to that which would be done in raising a weight equal to the sum of the weights through the height through which the C. G. of the weights has been lifted.*

Let W_1, W_2, W_3, ... be the several weights,
x_1, x_2, x_3, . . . the heights of their C. G. above a given plane;
and h the height of their C. G. above the same plane.
Then (Art. 196)—

$$(W_1+W_2+W_3+\ldots)h=W_1x_1+W_2x_2+W_3x_3+\ \ldots \quad (1.)$$

Now let the weights be lifted to heights k_1, k_2, k_3...above the given plane, and their C. G. to a height a above the same plane.

Then—

$$(W_1+W_2+W_3+\ldots)\,a=W_1k_1+W_2k_2+W_3k_3+\ .\ .\ (2.)$$

Subtracting (1) from (2) we have—

$$(W_1+W_2\ldots)\,(a-h)=W_1(k_1-x_1)+W_2(k_2-x_2)+\ .\ .\ .$$

Now the left-hand member of this equation represents the work done in lifting a weight equal to the sum of the weights through the height through which their C. G. is raised, and the right-hand member gives the total work expended in lifting the several weights; and thus the theorem is established. (Q.E.D.)

NOTE.—In transporting heavy bodies along a level railway, the principle of this theorem holds good, because in this case also the resistances overcome vary directly as the weights. (*See* Art. 259.)

EXAMPLES—LIX.

1. Find the work expended in digging a hole 16 feet deep, the area of section being 5 sq. feet, if the weight of a cubic foot of earth be 75 lbs.

2. Find the work expended in raising the materials for building a column 120 feet high and 15 feet square, if a cubic foot of the materials weigh 140 lbs.

3. A shaft whose diameter is 14 feet is sunk through 600 feet of chalk; find the work done in raising the materials, the mass of a cubic foot of chalk being 145 lbs.

4. In what time would an engine of 20 H.P. empty a shaft filled with water, the depth being 400 feet, and the diameter of the shaft 7 feet?

5. A railway is 120 miles long; every year 3000 tons of rails are distributed equally along the line. How much work is expended in transporting the metal from a store at one end?

6. How many journeys of 120 miles, performed by an engine of 40 tons, will the work expended in Ex. 5 represent, the resistance amounting to 10 lbs. weight per ton?

7. At what rate can an engine of 30 H.P. draw a train of 50 tons up an inclined plane rising 1 in 280, the resistance amounting to 7 lbs. weight per ton?

8. A horse drawing a wagon at the rate of 2 miles per hour exerts a traction of 154 lbs. weight; find the work done per minute.

9. A train is kept going on a level railroad at the uniform speed of 45 miles an hour, the resistances amounting to the weight of $1\frac{1}{2}$ tons. What power is the engine developing?

10. A rope, whose mass is 3 lbs. per fathom and whose length is 120 feet, hangs freely over the edge of a cliff. What work is done in hauling the rope to the top of the cliff?

11. A cable, consisting of 1000 links, each 6 inches long and weighing 4 lbs., hangs vertically from a viaduct; find the work done in raising the cable to the roadway.

12. The curtain of a stage is 30 feet long, 20 feet high, and weighs $\frac{3}{4}$ lb. per square foot; find the work done in gathering it to the level of its upper edge.

13. Five engines, whose combined H.P. is 1310, pump water from a depth of 73 fathoms; find the number of gallons raised per hour.

Note.—A gallon of water = 10 lbs.

14. Twenty men, by means of a treadmill, pump water to a height of 40 feet. In what time will they raise 10,000 gallons, supposing that one-third of the work is useless through friction, and that each man yields 3900 units of work per minute?

15. A coal-pit 1200 feet deep is flooded by a feeder discharging 900 gallons per minute at the bottom; if the united pumping power of the engines be 400 H.P., but the engines are in such bad condition that only 60 per cent. of their nominal H.P. is useful, what will be the result of the work?

16. An engine of 10 H.P., working 10 hours a day, supplies 2500 houses with water, which it raises to a mean level of 50 feet; required the average supply to each house. (A gallon of water = 10 lbs.)

309. When strict accuracy is necessary, as in estimating the Work done by an electric current, Work must be measured in terms of the Absolute or Kinetic Unit of Work. (*See* Art. 302.)

310. Def. *The* **Absolute unit of Work** *is the work done by the Kinetic unit of force exerted through the unit of space.*

The British Absolute Unit of Work is therefore the work done by a Poundal exerted through a foot.

This unit of work is called a '*Foot-poundal.*'

In the C.G.S. system, the Absolute Unit of Work is the work done by a Dyne exerted through a centimetre.

This Unit of Work is called an '*Erg.*'

311. If a force of F poundals be exerted through a distance of s feet, then—

The number of Absolute Units of work expended $=Fs$ *foot-poundals.*

NOTE.—If the force be expressed in Dynes, and the distance in centimetres, then the work done $=Fs$ *ergs.*

Example vii.—Find in absolute units the work done in raising a mass of 10 lbs. through a height of 20 feet at Greenwich, where $g=32{\cdot}191$.

$$\text{Work done}=Fs=10\times 32{\cdot}191\times 20$$
$$=6438{\cdot}2 \text{ ft.-poundals.}$$

If the mass were raised in the island of Trinidad, where $g=32{\cdot}091$, the work expended would be 6418·2 foot-poundals.

312. DEF.—**Energy** *is the capacity to do work.* A moving body must have a power to overcome resistance through some space, therefore a moving body has the capacity of doing work, and therefore possesses Energy.

313. *Let a body whose mass is m, having started with a velocity u, be acted on by a force F, and having described a space s, let its velocity be v; to find an expression for the work done.*

We know by Art. 63 that $v^2=u^2\pm 2as$;

$$\therefore\ mv^2=mu^2\pm 2mas;$$

but $ma=F$. (Art. 124, 128.)

$$\therefore\ \tfrac{1}{2}mv^2=\tfrac{1}{2}mu^2\pm Fs;\quad \therefore\ Fs=\pm(\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2).$$

Now Fs denotes the work done by the force F acting on the body while the body describes the space s. If we call the quantity $\frac{1}{2}mv^2$ (*i.e.* half the mass multiplied by the square of the velocity) by the name Kinetic Energy, we may draw the following conclusions from the equation now established—

(1) *If F act in the direction of the motion of the body*, we must take the positive sign;

$$\therefore \text{Work done} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2;$$

therefore Work done is measured by the *increase* of the Kinetic Energy of the body.

We infer, therefore, that the Kinetic Energy developed is the proper measure of the work done by any force which is wholly expended in producing motion.

(2) *If F act in the direction opposite to that of the motion of the body*, then we must take the negative sign;

$$\therefore \text{Work done} = \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2.$$

The force in this case, instead of doing work on the body, acts as a resistance, which the body in its motion overcomes. Therefore the Work done has been done at the expense of the Kinetic Energy which has been transformed.

A moving body can do work in overcoming resistance, and the Work done is measured by the *decrease* of the Kinetic Energy.

(3) *If the body start from rest*, then $u=0$;

$$\therefore \text{Work done} = \tfrac{1}{2}mv^2,$$

i.e. Work done is measured by the Kinetic Energy developed.

(4) *If the body is brought to rest*, then $v=0$;

$$\therefore \text{Work done} = \tfrac{1}{2}mu^2,$$

i.e. the Work done is measured by the Kinetic Energy which the body had when the resistance began to act.

We may see now the reason why the quantity $\frac{1}{2}mv^2$ is appropriately called the Kinetic Energy* of the body.

The **Energy** of a body is its capacity or power of doing work. This capacity may be due to several causes, such as the body's motion, position, heat, electrical state, and so on.

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* The term *Energy*, to denote in a precise sense the quantity of Work which a system can do, was introduced by Dr. Young.

The **Kinetic Energy** of a moving body is the power it has of doing work in virtue of its motion. This power of doing work is exactly measured at any instant by $\frac{1}{2}mv^2$, where v is the velocity at that instant.

Therefore, if the Kinetic Energy of a body be denoted by E, we can say that

$$E=\tfrac{1}{2}mv^2.$$

314. When a body is at rest at a height above the ground *it has a power of doing work* owing to the force of the earth's pull on it, and we infer from our definition of Energy that it has energy of some sort. Now Energy due to a body's position, relative to the earth or otherwise, has been variously termed 'Possible Energy,' 'Energy of Position,' 'Energy in Store,' and finally '**Potential Energy**' by Professor Rankine. This last is the term in most familiar use.

315. The work expended in raising a mass of m lbs. to a height of s feet $=mgs$. foot-poundals. (Art. 311.)

Now if the mass be let fall from the height s, and its velocity on reaching the ground be v, the Work done by gravity on the body will be measured by the Kinetic Energy acquired by the body, *i.e.* by $\frac{1}{2}mv^2$ foot-poundals. [Art. 313 (3).] Now $v^2=2gs$;

$\therefore$ Work done by gravity $=\frac{1}{2}m\times 2gs=mgs$ foot-poundals.

So that the body by its fall acquires sufficient energy to *reproduce the work expended in raising it to the height from which it fell.*

316. Thus none of the work expended in raising a body to any height is lost, but is all stored up in the body. If the body be let fall, it gradually loses its Potential Energy and acquires Kinetic Energy, until on returning to the ground, having parted with all its Potential Energy, it is in possession once more of an equivalent amount of Kinetic Energy.

A numerical example may make this somewhat clearer.

If a mass of 10 lbs. be raised to a height of 144 feet at a place where $g=32{\cdot}2$, the work expended $=10\times32{\cdot}2\times144$ $=46368$ foot-poundals.

To show that this amount of work is stored up in the body in its new position—

If let fall, it will strike the ground with a

$$\text{Velocity}=\sqrt{64{\cdot}4\times144}\text{ f.-s.}$$

$$\therefore\text{ its K.E.}=\tfrac{1}{2}\times10\times64{\cdot}4\times144=46368\text{ foot-poundals.}$$

Thus the K. E. gained is equal to the P. E. lost.

And it may be easily shown that, at *any* point of the body's ascent or descent, the sum of the K. E. and P. E. is 46368 foot-poundals.

317. The following are instances of bodies possessing *Potential Energy*:—A stone resting on a wall, or imbedded in a vertical cliff—A bent crossbow—A bent spring—Compressed air—The weights of a clock when wound.

The following are cases of bodies possessing *Kinetic Energy*:—A projectile when fired—Air in motion—Water in motion—The Earth moving through space.

The following may be cited as *Stores of Energy*:—A head of water —The tides on a coast—An electric battery—Food—Coal and wood used as fuel—A charge of gunpowder.

318. Let us next consider the case where, apparently, Kinetic Energy is lost and no equivalent is obtained; for instance, when a ball falls dead on a hard surface. We may ask what has become of the Energy, because seemingly no work has been done. Recent advances in physical science enable us to answer that the Kinetic Energy has been transformed into another kind of Energy, which we name *Heat*.

319. Dr. Joule's experiments have established the extremely important fact that the amount of heat capable of raising the temperature of a pound of water from 60° to 61° F. corresponds to 772·43 foot-pounds of work.

In other words, if 772·43 lbs. be allowed to fall through a foot, and the work done in the descent be exclusively used in imparting heat to a pound of water by a rotating paddle, or otherwise, the result will be that the temperature of that amount of water will be raised from 60° to 61° F.

The Heat of a body is explained as being the vibratory motion of its molecules through very small spaces.

320. The above is a case of the **Transformation of Energy.**

The following additional illustrations may be given :—

1. When a hammer strikes an anvil, its energy is changed into Heat.

2. When a train has the steam shut off, and the brakes bring the train to rest on a level railroad, the whole of the K. E. has been changed into Heat.

3. If the train had gone a certain distance up a slope before the brakes stopped it, the K. E. would be changed partly into Potential Energy and partly into Heat.

321. The case of a body being raised and thus becoming possessed of Potential Energy, and falling and thus acquiring its original Kinetic Energy (Arts. 315, 316), is a simple case of an extremely important modern generalisation known as the **Conservation of Energy.**

This may be stated as follows :—

'The total Energy of any system is a quantity which can neither be increased nor diminished by any action between the parts of that system, though it may be transformed into any of the forms of which Energy is susceptible.'

322. Dr. Joule proved the law according to which Work may be changed into Heat. Sir W. Thomson and others have given attention to the law by which Heat may be changed back again into Work. But there is a most important difference between these laws. While all the Work expended can be transformed into Heat, by no method can all the Heat be changed back again into Work.

If a hammer strike a nail, we cannot recover the heat evolved by the blow.

Now no machine can be made to move without friction; and from this follows the impossibility of ever constructing a piece of mechanism which will go for ever: in other words, perpetual motion is a thing impossible. The Kinetic Energy of any machine left to itself will eventually be transformed into the Energy of Heat.

323. Let this consideration be also applied to the system of which the sun is the source of energy. The sun's rays have given us vast stores of energy in various forms, such as Fuel, Tides, animal and vegetable Food, and so on. From the sun's energy in the past and in the present are thus derived the activities of our earth. The energy given out by the sun is estimated at 500 H. P. from every square yard of its surface. If the sun continue to give out more heat than it receives, it must become cooler in process of time. If the high temperature be due to the shrinking of its mass, there must come a time when the universe will become an equally heated mass, and, of course, not then adapted for the production of Work, since such depends on the difference of temperature.

The student is advised to read the late Professor B. Stewart's *Conservation of Energy*.

Example viii.—Find the energy of a body whose mass is 10 lbs., and moving with a velocity of 30 f.-s.

$$E=\tfrac{1}{2}mv^2=\tfrac{1}{2}\times 10\times 30^2=4500 \text{ foot-poundals.}$$

Example ix.—A projectile travelling at the rate of 700 f.-s. penetrates to the depth of 2 inches; find the velocity necessary to penetrate 3 inches.

Let F be the resistance offered to the penetration.

Then $Fs=\tfrac{1}{2}mv^2$.

In the first case, $F\times\tfrac{2}{12}=\tfrac{1}{2}m\,(700)^2$.

In the second, $F\times\tfrac{3}{12}=\tfrac{1}{2}mv^2$;

$$\therefore \frac{v^2}{(700)^2}=\frac{3}{2};$$

From this equation, $v=857{\cdot}4$ f.-s.

The following Examples will show how the 'Principle of Work' may be applied to solve many questions:—

Example x.—A mass of 8 lbs. is rolled on grass with a velocity of 12 f.-s. If the resistance be $\frac{1}{10}$th of the weight, how far will the body move?

$$Fs = \tfrac{1}{2}mv^2;$$
$$\therefore \tfrac{1}{10} \times 8g \times s = \tfrac{1}{2} \times 8 \times 12^2;$$
$$\therefore s = 22\tfrac{1}{2} \text{ feet.}$$

Example xi.—The last carriage of a train gets loose while the train is running at the rate of 30 miles an hour up an incline of 1 in 150. If the resistance from friction amount to $\frac{1}{300}$ of the weight of the carriage, how far will the carriage ascend the slope?

Let m = mass of the carriage;
$\therefore$ mg = weight ,,
and 44 f.-s. = velocity ,,
Then $Fs = \frac{1}{2}mv^2$;

$$\therefore \left(\frac{mg}{300} + mg\sin i\right) \times s = \tfrac{1}{2}mv^2;$$
$$\therefore \left(\frac{32}{300} + \frac{32}{150}\right) \times s = \tfrac{1}{2} \times 44^2.$$

From this equation, $s = 3025$ feet.

Example xii.—Through what space must a force equal to $\frac{1}{2}$ lb. weight act on a mass of 48 lbs. to increase its velocity from 24 f.-s. to 36 f.-s.?

$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$
$$\therefore \tfrac{1}{2}g \times s = \tfrac{1}{2} \times 48 \times 36^2 - \tfrac{1}{2} \times 48 \times 24^2;$$

From this equation, $s = 1080$ feet.

EXAMPLES—LX.

1. What is the energy of a mass of 10 lbs. moving with a velocity of 10 f.-s.?

2. What energy does a mass of 300 lbs. possess if its velocity be 60 f.-s.?

3. Express in foot-pounds the energy stored up in a mass of 30 lbs. resting on the edge of a cliff 140 feet high.

4. By how much is the energy of a hundredweight of matter increased when raised vertically through 10 feet?

5. Find the increase of energy when 56 lbs. are raised along a smooth plane 300 feet long and inclined at 30° to the horizon.

6. A projectile whose mass is 64 lbs. leaves a gun with a muzzle velocity of 1200 f.-s.; what work is stored up in it at that instant?

7. The mass of a cutter is $2\frac{1}{2}$ tons; she carries 13 men weighing on an average $10\frac{1}{2}$ stones; find her energy when moving with a speed of 4 miles an hour.

8. A mass of 6 lbs. falls freely from rest; by how much is its kinetic energy increased between the 3d and the 8th seconds?

9. Find the work done by its weight on a mass of 8 lbs. falling from rest at a height of 10 feet.

10. A mass of 10 lbs. falls for 6 seconds; what work has been done on it by its weight?

11. Two masses of 10 lbs. and 6 lbs. respectively are connected by a light string over a smooth peg; what work is done by gravity on the system in 4 seconds?

12. Two masses of 3 lbs. and 2 lbs. respectively are connected by a light string over a smooth peg; find the work done on the system by gravity in 5 seconds.

13. In Ex. 11, what work can the 10 lbs. do after motion has lasted for 8 seconds?

14. What work (in foot-pounds) is stored in a tank 100 feet long, 50 feet wide, and 8 feet deep when full, the base being raised 18 feet from the ground? (A cubic foot of water $=62\frac{1}{2}$ lbs.)

15. In pile-driving 40 men raise a hammer 15 times in an hour, its mass being 12 cwt., and the height to which it is raised is 110 feet; find the work done by each man in a minute.

MISCELLANEOUS EXAMPLES—LXI.

1. A body whose mass is 25 lbs., moving with a velocity of 30 f.-s., meets a constant resistance equal to the weight of 4 lbs.; how far will the body move?

2. A ball lies on the deck of a ship, and is observed to roll backwards 25 inches; if the friction between the ball and the deck be $\frac{1}{16}$ of the weight of the ball, find how much the velocity of the ship has changed.

3. If two bodies, moving with the same velocity, possess between them x units of work, show that if their masses be m and m_1, the number of units of work accumulated in m is $\frac{mx}{m+m_1}$.

4. When a particle slides down a smooth inclined plane, show that it loses none of its energy.

5. If a projectile of mass 64 lbs., and moving with a velocity of 1200 f.-s., penetrate 4 inches into a target, find the resistance.

6. A body slides down a plane whose height is 15 feet, and length 25 feet; how far will it slide along a horizontal plane at the foot, if the coefficient of friction for both surfaces be $\frac{1}{7}$, and no velocity be lost on passing from the slope to the level?

7. The mean pressure in a steam cylinder is 30 lbs. weight per square inch, the area of the piston is 1000 square inches, the stroke is 3 feet, and 45 double strokes are made per minute; find the H.P. of the engine.

8. A body runs from rest down an incline of 1 in 100 for one mile; how far will it be carried along the level at the foot of the incline, supposing the resistance to be 8 lbs. weight per ton for both surfaces.

9. Establish the equation $v^2 = u^2 + 2gs$ for the motion of a falling body, and express it in the language of the Science of Energy.

10. A ball of 10 lbs. is dropped from a height of 289·8 feet, but, after falling half-way, it explodes, dividing into two equal parts, one of which is reduced to rest by the explosion; find the subsequent motion of each part, and determine the Kinetic Energy developed by the explosion. ($g = 32{\cdot}2$.)

11. Define the *work done* by a machine. In what units is it measured?

12. Show that the work done in drawing a load up a rough inclined plane is equal to that done in drawing it along the horizontal base, supposed to be equally rough, and then raising it vertically up the height.

13. Define the terms *work*, *energy*, *horse-power*.

14. If one yard, one minute, and one ton be the units of length, time, and mass respectively, express the unit of work in foot-pounds. Find also in the given units the Kinetic Energy possessed by a mass of 1 cwt., moving with a velocity of 10 f.-s.

15. Prove that if M be the mass of a gun and its carriage, and m the mass of the shot, the Energy of recoil : Energy of the shot :: m : M.

16. A 700 lbs. shot issues with a velocity of 1600 f.-s. from a 35-ton gun; find (1) the energy of translation of the shot; (2) the energy of the gun's recoil.

17. The resistance to the motion of a train on the level is 12 lbs. weight for every ton, and it is found that the maximum speed of a train of 100 tons with a given engine is 50 miles an hour. Trucks whose mass is 20 tons are added to the train; show that the greatest speed then obtainable up an incline of 1 in 150 is a little greater than 18½ miles an hour, the friction remaining the same.

18. Define Kinetic Energy, and explain what is meant by Potential Energy.

19. What is meant by an 'Erg'?

Having given that a foot is 30·4797 centimetres, and that a pound is 453·59 grammes, express an 'erg' in British units.

20. If a body be acquiring Kinetic Energy, and not losing any other kind of energy, what inference may be drawn?

21. The Kinetic Energy of a raindrop in increased fourfold, while its momentum has increased threefold; in what ratios have its velocity and its mass increased?

22. How many cubic feet of water will an engine of 100 H.P. raise in one hour from a depth of 150 feet, if the efficient work be half the work applied, the mass of a cubic foot of water being 62·5 pounds?

23. Apply the principle of Transformation of Energy to solve the following problem: A train is moving on a horizontal railway at the rate of 30 miles an hour; if the steam be suddenly turned off, how far will it run before it stops, the resistances being taken as equal to the weight of 15 lbs. per ton?

24. Find the work done, in foot-tons, in order to propel a 256 lbs. shot with a uniform velocity of 1400 f.-s.

25. Suppose that the shot in the last example loses 140 f.-s. in passing through the air, find the energy of the blow in foot-tons on reaching the target.

26. If the target, in this case, stop the shot in the space of 7 inches, find the mean resistance of the target.

27. What work is done by a winding engine which raises a cage, whose mass is 3 tons, from a pit 200 yards deep, and causes it to have a velocity of 24 f.-s. at the top?

28. A body whose mass is 56 lbs. is projected with a velocity of 40 f.-s. along a rough horizontal surface; how far will it move against a resistance of ¼ of its own weight?

29. A train whose mass is 100 tons is moving with a speed of 45 miles an hour; if the steam be now shut off, and the force of the brakes and the friction be a resistance of 30 lbs. weight per ton, how far will the train move on the level?

30. A mass of 112 lbs. is moved 60 feet along a rough level plane; if the resistance be $\frac{1}{14}$ weight of the body, find the work done against that resistance.

31. An engine draws a train of 90 tons up a smooth slope of 1 in 270 at a uniform speed of 30 miles an hour; at what rate is the engine working?

32. A train of 120 tons, while going a mile, increases its speed uniformly, and has then a velocity of 30 miles an hour; if the friction and the air offered a steady resistance of 15 lbs. weight per ton, find the H.P. of the engine.

33. A train is running with a speed of 45 miles an hour; how far will it move (1) going up a slope of 1 in 100, (2) going down the slope, if the brake-power be 60 lbs. weight per ton, and the steam be shut off?

34. An engine moves, from rest, a train of 60 tons a distance of 400 yards on a level against a resistance of $\frac{3}{4}$ ton weight, and gets up a velocity of 15 miles per hour; find the work done by the engine.

35. An engine moves, from rest, a train of 80 tons half a mile up a slope rising 1 in 240 against a resistance from friction of $1\frac{1}{2}$ tons weight, and has then a velocity of 15 miles an hour; find the work done by the engine.

36. A train is travelling with a speed of 30 miles an hour. If the steam be shut off and a brake-power of $\frac{1}{300}$ weight of the train be applied, how far will the train move, the coefficient of kinetic friction being $\frac{1}{8}$?

37. A train, whose mass is 90 tons, comes to the foot of a slope rising 1 in 160, and begins to move up with a velocity of 30 miles an hour. The resistance from friction is 7 lbs. weight per ton. The slope is 2 miles long. At the top the velocity is 20 miles per hour. Find (1) how many units of work have been expended by the engine; (2) how far the same work would have taken the train with uniform velocity on a level.

38. The diameter of a piston is $8\frac{1}{4}$ inches, the pressure of steam is 21 lbs. weight per square inch, the length of stroke 5 feet, and the number of double strokes is 18 per minute. It is found to raise 40 cubic feet of water per minute from a depth of 200 fathoms. What is its efficiency?

39. The energy of a charge of powder being 2050 million foot-poundals, find the velocity of a projectile of 1600 lbs. when fired, supposing that $\frac{4}{21}$ per cent. of the energy is transformed into heat and light.

40. A mass of m lbs. falls a feet, and drives a pile whose mass is $2m$ lbs. a distance x feet; find the resistance, there being no recoil.

41. A steam hammer of 10 tons is driven by a pressure of 30 lbs. weight per square inch acting on an area of 800 square inches. If the hammer descend 8 feet and compress a body 6 inches, what steady pressure has it exerted during the compression?

42. An engine running at 45 miles per hour comes to a water trough, and scoops up 800 gallons of water, which are raised to an average height of $8\frac{1}{4}$ feet. Neglecting all loss of energy by impact and friction, show that the work done is 610500 ft.-lbs. (1 gallon of water = 10 lbs.)

43. If 1 lb. of gunpowder can give out 90 ft.-tons of energy, when fired in the bore of a gun, prove that the velocity which will be imparted to a projectile of mass 100 lbs. by a charge of 20 lbs. will be 1608 f.-s. nearly.

44. A projectile of mass m lbs. is fired with a velocity of v f.-s. into the block of a ballistic pendulum of mass M lbs. Show that the energy of its motion degraded into heat during the impact is $\frac{Mmv^2}{2(M+m)g}$ ft.-lbs.

45. A battle-ship on her forced draught trials made $18\frac{3}{4}$ knots, the engines indicating 12900 H.P. Supposing that all the power is used in the propulsion, show that the resistance at this speed is nearly 100 tons weight. (See Art. 13.)

46. If an engine of x H.P. working at full power is drawing on the level a train of M tons at a speed of y f.-s. against a resistance of R lbs. weight per ton, prove that the *acceleration*, a, at that instant can be found from the equation

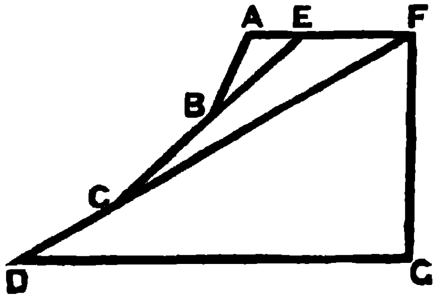
$$x \times 550\,g = (M.Rg + 2240\,Ma)\cdot y.$$

47. If in No. 46. $x = 160$, $M = 120$, $R = 10$, $y = 36\frac{2}{3}$, prove that $a = \frac{1}{6}$ f.-s.-s.

CHAPTER XVI.

THE PENDULUM.

324. *If a body fall down a series of inclined planes, its velocity at the bottom is equal to the velocity due to the entire vertical height, provided that it loses no velocity in passing from any plane to the next plane in the series.*



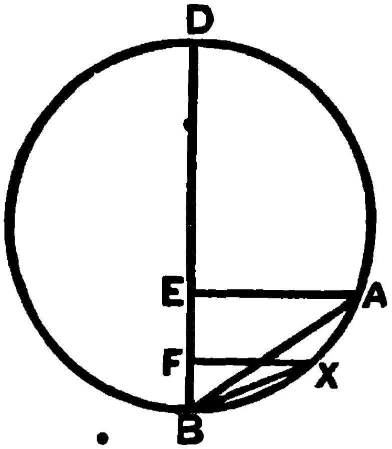
By Article 73, the Velocity at B = Velocity due to the height of the plane AB.

The Velocity at C = Velocity due to the height of the plane EC.

The Velocity at D = Velocity due to the height of the plane FD

= Velocity due to FG. (Q.E.D.)

325. *To find the Velocity acquired in falling down the arc of a circle.*



The arc of a circle may be considered as made up of an infinite number of small inclined planes.

By Article 324, the velocity acquired in falling from A to $X =$ velocity due to the vertical height EF.

Let velocity at $X = v$,

Then, $v^2 = 2g.EF$, ... (1.)

Let $a =$ chord of the arc BA; $x =$ chord of BX; and $r =$ radius of the circle.

By Euclid vi. 8, $DB.BE = BA^2$; and $DB.BF = BX^2$.

$$\therefore 2\,r.BE = a^2;\ 2r.BF = x^2.$$

$$\therefore BE = \frac{a^2}{2r};\ BF = \frac{x^2}{2r}.$$

$$\text{Now } EF = EB - FB,$$

$$\therefore EF = \frac{a^2 - x^2}{2r}.$$

Substituting in Equation (1) we have—

$$v^2 = 2g\left(\frac{a^2 - x^2}{2r}\right),$$

$$\therefore v = \sqrt{\frac{g}{r}\,(a^2 - x^2)}.$$

And this is the velocity acquired in falling down the arc AX.

326. A body suspended from a fixed point by a fine string or wire, and oscillating in a vertical plane, is called a **Simple Pendulum**.

The time of swinging from its highest point on one side of the vertical line through the point of suspension to its highest point on the other side is called the **Time of oscillation.**

The time of oscillation is also called a *beat*, or the *time of vibration*.

By some writers the whole interval between a body leaving the highest point and coming back to it again, is called the Time of oscillation.

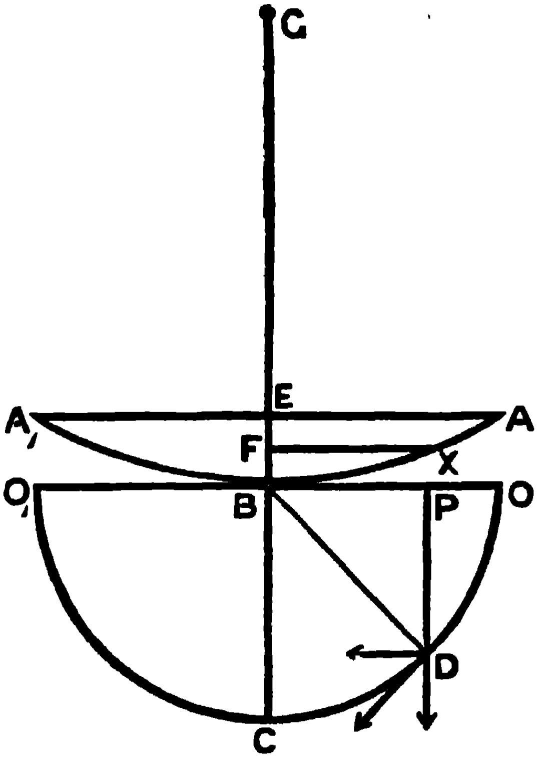
To find the Time of Oscillation of a Simple Pendulum.

327. Let a particle be suspended from a point G by a string GB without weight, and let it describe a *small* arc AA_1 such that its length does not sensibly differ from that of its chord.

With B, the lowest point of the arc, as centre, and radius $= BA$, describe a semicircle OCO_1 cutting the tangent at B in the points O and O_1.

The velocity of the particle at A is zero.

Using notation of Art. 325, the velocity of the particle at any point $X = \sqrt{\frac{g}{r}(a^2 - x^2)}$.



Make $BP = BX$. From P draw the vertical line PD, cutting the semicircle in D. Join BD.

Now P will describe OBO_1 as X describes ABA_1.

$\therefore$ Velocity of $P =$ Velocity of X.

The *Horizontal* Velocity of the point D = Velocity of P

$$\therefore \text{ Velocity of } D = \text{Velocity of } P \times \sec PDB$$

$$= \sqrt{\frac{g}{r}(a^2-x^2)} \times \frac{BD}{PD} = \sqrt{\frac{g}{r}(a^2-x^2)} \times \frac{a}{\sqrt{a^2-x^2}}$$

$$= a\sqrt{\frac{g}{r}} \text{ which is constant.}$$

Therefore X describes the arc ABA_1, as D describes the semicircle with the uniform velocity $a\sqrt{\frac{g}{r}}$.

328. *To find time of describing* OCO_1.

We know that $s = vt$;

$$\therefore \pi a = a\sqrt{\frac{g}{r}} \cdot t;$$

$$\therefore t = \pi\sqrt{\frac{r}{g}};$$

$$\therefore \text{ Time of describing } ABA_1 = \pi\sqrt{\frac{r}{g}}.$$

If we denote the length of the pendulum by l, and the time of oscillation by t, we have

$$t = \pi\sqrt{\frac{l}{g}}.$$

It will be noticed that the time of oscillation depends only on the *length of the pendulum*, and the *acceleration due to gravity* at the place.

If a pendulum of known length (l feet) be made to oscillate some hundreds of times at any place, and the whole time of doing so (in seconds) be noted, we can find the value of a single oscillation, *i.e.* the value of t, very accurately. Hence the local value of g can be found by this equation.

329. When the arc is small the length of the arc does not appear in the formula; we infer, therefore, that for small

arcs the time of oscillation is independent of the length of the arc. From this property the oscillations are said to be *isochronous.*

It will be also noticed that if the pendulum be shortened or the value of g be increased, the time of oscillation is diminished.

Example i.—Find the time of oscillation of a pendulum 48 inches long at a place, where $g = 32 \cdot 139$.

$$t = 3 \cdot 141 \sqrt{\frac{48}{32 \cdot 139 \times 12}}.$$

From which equation, $t = 1 \cdot 108$ seconds.

Example ii.—If the time of oscillation of a pendulum 25 inches long at Paris be $\cdot 799$ sec., find the value of g at Paris.

$$\cdot 799 = 3 \cdot 141 \sqrt{\frac{25}{g \times 12}}.$$

From which equation, $g = 32 \cdot 18$.

330. *To find the length of a Seconds Pendulum at a place where the acceleration due to gravity is g.*

By Article 328, $t = \pi \sqrt{\frac{l}{g}}$.

Let $t = 1$ second ;

$$\therefore 1 = \pi \sqrt{\frac{l}{g}} ;$$

$$\therefore l = \frac{g}{\pi^2} .$$

Example i.—Find the length of a seconds pendulum at a place, where $g = 32 \cdot 19$.

Let length $= l$ inches.

Using the Formula just established.

$$l = \frac{32 \cdot 18 \times 12}{(3 \cdot 141)^2}.$$

From which equation, $l = 39 \cdot 13$ inches.

331. *To find the number of oscillations gained or lost at any place in a given time when the length of the Pendulum is slightly altered.*

Let a pendulum of length l make n beats in T seconds.

" " $l+a$ " n_1 " "

Then, Times of Oscillation $=\frac{T}{n}$ and $\frac{T}{n_1}$ respectively.

By Article 328, $\frac{T}{n}=\pi\sqrt{\frac{l}{g}}$; $\frac{T}{n_1}=\pi\sqrt{\frac{l+a}{g}}$;

$$\therefore \frac{n_1}{n}=\sqrt{\frac{l}{l+a}};$$

$$\therefore 1-\frac{n_1}{n}=1-\sqrt{\frac{l}{l+a}};$$

$$\therefore \frac{n-n_1}{n}=1-\left(1+\frac{a}{l}\right)^{-\frac{1}{2}};$$

but $\left(1+\frac{a}{l}\right)^{-\frac{1}{2}}=1-\frac{a}{2l}+$ other terms which may be neglected;

$$\therefore \frac{n-n_1}{n}=1-1+\frac{a}{2l};$$

$$\therefore n-n_1=\frac{na}{2l};$$

and $(n-n_1)$ is the number of beats gained or lost in the time T by shortening or lengthening the pendulum.

Example i.—The pendulum which beats seconds at Greenwich is 39·139 inches long. If its length be shortened by two turns of a screw which makes 32 turns in an inch, how much will the pendulum gain in a week?

Here, $n=7\times 86{,}400$ seconds

$a=\frac{1}{16}$ in.

$l=39{\cdot}139$.

$$\text{Then, } n-n_1=\frac{7\times 86{,}400\times\frac{1}{16}}{2\times 39{\cdot}139}.$$

From which equation, $(n-n_1)=483$ beats gained.

Example ii.—A pendulum supposed to beat seconds at Greenwich loses 20 seconds a day; find the error in its length.

Here, $n-n_1=20$; $n=86{,}400$; $l=39{\cdot}139$;

$$\therefore\ a=\frac{2l\,(n-n_1)}{n}=\frac{2\times 39{\cdot}139\times 20}{86{,}400}.$$

From which equation, $a={\cdot}0181$.

Therefore the pendulum is *too long* by ·0181 inch.

332. *To find the number of oscillations gained or lost in a given time when the instrument is taken from one place to another.*

Let g and g_1 be the accelerations due to gravity at the two places.

As in Article 331, $\frac{T}{n}=\pi\sqrt{\frac{l}{g}}$; $\frac{T}{n_1}=\pi\sqrt{\frac{l}{g_1}}$;

$$\therefore\ \frac{n_1}{n}=\sqrt{\frac{g_1}{g}};$$

$$\therefore\ \frac{n_1^2}{n^2}=\frac{g_1}{g};$$

$$\frac{n^2-n_1^2}{n^2}=\frac{g-g_1}{g};$$

$$\therefore\ \frac{(n+n_1)\,(n-n_1)}{n^2}=\frac{g-g_1}{g};$$

but, $n+n_1=2n$, nearly.

$$\therefore\ \frac{2n\,(n-n_1)}{n^2}=\frac{g-g_1}{g};$$

$$\therefore\ n-n_1=\frac{n\,(g-g_1)}{2g}.$$

and $(n-n_1)$ is the number of oscillations gained or lost.

From this result may be deduced the solution of the converse problem, viz.:—

333. *To compare the acceleration due to gravity at two places by observing the number of oscillations at the places in a given time.*

Example i.—A pendulum beating seconds at Greenwich, where $g=32{\cdot}19$, is carried to the Equator, where $g=32{\cdot}09$; how much does it lose in a day?

$$\text{Here, } n=86{,}400;\quad g=32{\cdot}19;\quad g_1=32{\cdot}09;$$

$$\therefore n-n_1=\frac{86{,}400\,(32{\cdot}19-32{\cdot}09)}{2\times 32{\cdot}19}.$$

From which equation, $n-n_1=134$ seconds.

Example ii.—A pendulum beat 85,945 times at Greenwich in a mean solar day, and when taken to Paris it was found to beat 85,933 times; if $g=32\ 19$ at Greenwich, find its value at Paris.

$$\text{Here, } n-n_1=12;$$

$$\therefore 12=\frac{85{,}945\,(g-g_1)}{2\times 32{\cdot}19}.$$

$$\text{From which equation, } g-g_1={\cdot}00899;$$

$$\therefore g_1=32{\cdot}19-{\cdot}0089$$

$$=32{\cdot}181 \text{ at Paris.}$$

334. *To find the Height of a Mountain by observing the number of oscillations lost in a given time.*

NOTE.—The acceleration at any point due to gravity *varies inversely as the square of the distance from the centre* of the earth when the point is *outside* the earth.

Let a pendulum whose length is l beat n times at the sea-level in T seconds, and n_1 times at the top of a mountain in the same time.

Let g=acceleration at the sea-level;
Let g_1= „ „ summit;
Let h= height of mountain in miles.

$$\text{Then, } g_1 : g=\frac{1}{(r+h)^2}:\frac{1}{r^2};$$

$$\therefore \frac{g_1}{g}=\frac{r^2}{(r+h)^2}.$$

$$\text{But, } \frac{n_1}{n}=\sqrt{\frac{g_1}{g}};\qquad \text{(Art. 332.)}$$

$$\therefore \frac{n_1}{n}=\frac{r}{r+h}.$$

From which equation, $h=\dfrac{r\,(n-n_1)}{n_1}=\dfrac{r\,(n-n_1)}{n}$, nearly.

Example.—A pendulum beating seconds at the foot of a mountain is found to lose 50 seconds a day when taken to the top. Find the height of the mountain, on the assumption that the earth is a sphere whose radius = 4000 miles.

$$h = \frac{4000 \times 50}{86,400}.$$

From which equation, $h = 2 \cdot 31$ miles.

EXAMPLES—LXII.

1. Find the length of a seconds pendulum in Spitzbergen, where $g = 32 \cdot 2528$.

2. Find the length of a seconds pendulum at Stockholm, where $g = 32 \cdot 212$.

3. Find the length of a pendulum which beats seconds in the Island of Ascension, where $g = 32 \cdot 095$.

4. If a pendulum oscillate once in two seconds in London, where $g = 32 \cdot 19$, find its length.

5. Find the length of a pendulum which beats half-seconds at Greenwich. ($g = 32 \cdot 19$.)

6. A clock loses two minutes in a day; how many turns must be made on a screw to correct the error, if there be 50 turns in an inch?

7. A seconds pendulum at Greenwich is 39·139 inches long; if a clock gain 58 seconds in a day, how much is the pendulum out?

8. The length of the seconds pendulum at the Equator is 39·014 inches; find the local acceleration caused by gravity.

9. If the length of the seconds pendulum at Greenwich be 39·139, find the length of a pendulum the time of oscillation of which is 2 seconds.

10. Find the number of seconds lost by a clock in a day at the bottom of a mine 2400 feet deep.

11. A pendulum which beats seconds at Greenwich gains 16 beats in a day at a place A; find the value of g at A, the value of g at Greenwich being 32·19.

12. Find the length of a pendulum which at Greenwich beats 3720 in every hour. ($g = 32 \cdot 19$.)

13. A clock loses four minutes a day; how much is the pendulum out?

14. A clock gains $2\frac{1}{2}$ minutes a day; find the error in the length of its pendulum.

15. If the length of a pendulum which beats seconds in Mauritius be 39·0468 inches, find the local value of g.

16. A clock loses five seconds in a day; how much must its pendulum be shortened, if its length be 39·139 inches?

17. What is the length of a pendulum which beats 240 times a minute at Greenwich? ($g=32{\cdot}19$.)

18. Suppose that a pendulum, which at the Equator beats seconds, would, if carried to the Pole, gain five minutes a day; compare the values of g at the two places.

19. If l be the length of a seconds pendulum at a place, and l_1 the length of a pendulum which beats once in n seconds at the same place, prove that $l_1=n^2l$.

20. At a place where the length of a seconds pendulum is 39·047 inches, find the time occupied by 100 beats of a pendulum 11 feet long.

21. A pendulum gains 36 seconds a day; how high must it be raised above the sea-level in order that it may keep correct time? (Radius of earth = 4000 miles.)

22. A seconds pendulum is found to lose 48·6 seconds in a day at the summit of a mountain; find the height of the mountain.

23. A seconds pendulum (= 39·139 inches) is lengthened by 1 05 inches; find the time lost in a day.

24. At what height will a seconds pendulum beat 3588 times in an hour, the earth's radius being taken as 3958 miles?

25. Two pendulums whose lengths are l and l_1 respectively begin to oscillate at the same instant, and after x oscillations they beat together again; if l, the greater, be known, find l_1.

26. A seconds pendulum at one place when taken to another place loses x minutes a day. It is made to keep correct time by shortening it by a inches; find the length of the pendulum.

27. A pendulum 36 feet long made 1662 oscillations in 500 seconds; find the local value of g.

28. A pendulum makes n oscillations at A in the same time that it makes n_1 oscillations at B. If a string can support a mass m at A, what mass can it support at B?

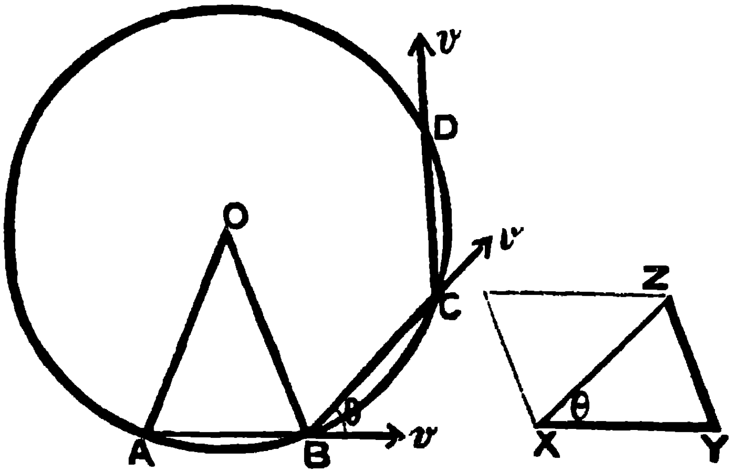
Example.—Let $n=640$; $n_1=641$; $m=100$.

29. Given the length of a seconds pendulum at Greenwich = 39·139 inches, find the length of the pendulum which oscillates nine times in a minute at that place.

30. Find the length of a pendulum which oscillates as often in a minute as there are inches in its length.

31. Find the length of a pendulum which oscillates three times during the time that a particle falls through 81 feet.

335. *If a particle describe a circle with uniform velocity, to find the direction and magnitude of the acceleration.*



Let *ABCD* . . . be a regular polygon of *n* sides inscribed in a circle.

Let τ be the time of describing each of the equal sides *AB*, *BC*, . . . when the uniform velocity $=v$;

$$\therefore AB = v\tau.$$

Draw *XY*, *XZ* to represent in magnitude and direction the velocities of the particle at *B* and *C* respectively.

Then *YZ* will represent the change of velocity in the time τ. (Article 32.)

To find the direction of YZ.

Let the angle $YXZ=\theta=$angle between *BC* and *AB* produced.

$$\text{Now the angle } YZX=\frac{\pi}{2}-\frac{\theta}{2},$$

$$\text{And the angle } OBC=\tfrac{1}{2}(\pi-\theta)=\frac{\pi}{2}-\frac{\theta}{2},$$

$$\therefore \text{the angle } YZX=OBC.$$

If XZ be placed on BC, we shall have YZ parallel to BO;
∴ Direction of the 'change of velocity' is towards the centre of the circle.

To find the magnitude of the acceleration.

Let a be the measure of the acceleration.

We have seen that YZ represents the change of velocity in the time τ.

$$\therefore\ YZ = a\tau;$$

Now $XY = XZ = v$; and $n\theta = 2\pi$. (Euc. i. 32, Cor.)

$$\text{But } \tfrac{1}{2} YZ = XZ \sin \tfrac{1}{2}\theta;$$

$$\therefore\ YZ = 2XZ \sin \frac{\pi}{n};$$

$$\text{Also } YZ = a\tau;$$

$$\therefore\ a\tau = 2v \sin \frac{\pi}{n} \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Again, } \frac{AB}{2} = OB \sin \tfrac{1}{2} AOB;$$

$$\therefore\ AB = 2r \sin \tfrac{1}{2} \cdot \frac{2\pi}{n};$$

$$\therefore\ v\tau = 2r \sin \frac{\pi}{n} \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2) we have—

$$\frac{a}{v} = \frac{v}{r};$$

$$\therefore\ a = \frac{v^2}{r}.$$

Now when the sides of the polygon are infinitely small, we have the case of a body describing a circle with uniform velocity.

Therefore we infer that when a body describes a circle with uniform velocity, the force which compels it to take this path is always directed towards the centre of the circle, and produces an acceleration in the same direction whose measure is $\frac{v^2}{r}$. (*See* Art. 46.)

Hence if the Mass of the Body $= m$ lbs.,

The Measure of the Force $= \frac{mv^2}{r}$ poundals. (*See* Art. 111.)

Example.—A mass of 10 lbs. is attached to one end of a string 14 feet long, the other end being fixed, and is made to describe the circle twice in a second; find the tension of the string.

$$v \times 1 = 2 \times 2\pi \times 14;$$

$$\therefore\ v = 56 \times \frac{22}{7} = 176 \text{ f.-s.}$$

$$F = \frac{mv^2}{r} = \frac{10 \times (176)^2}{14 \times 32} \text{ lbs. weight.}$$

From which equation, $F = 691\frac{3}{7}$ lbs. weight.

EXAMPLES—LXIII.

1. A tramcar whose mass is $1\frac{1}{2}$ tons is travelling at the rate of 5 miles an hour, and goes round a curve whose radius is 220 yards; find the outward pressure on the rails.

2. A carriage whose mass is 4 tons, and moving at the rate of 20 miles an hour, goes round a curve whose radius is 1100 feet; find the outward pressure against the rails.

3. A carriage moving at the rate of 30 miles an hour goes round a curve the radius of which is 1600 feet. A mass of 8 lbs. hangs by a string from the roof; what horizontal force will keep the string vertical while the train is rounding the curve?

4. A mass of 10 lbs. is fixed at the end of a wire 4 feet long, and made to revolve 100 times a minute in a circle; find the tension of the wire.

5. A point describes the circumference of a circle whose radius is 16 feet with a velocity of 15 f.-s.; find the acceleration.

6. The width between the rails is $56\frac{1}{2}$ inches; how much is the outer rail higher than the inner on a curve of 1500 feet radius, when the speed on the line is 30 miles an hour?

7. At what angle (α) must the 'way' be inclined, so that the pressure of an engine passing along a curve of given radius (r) at a given speed (v) may be perpendicular to the ground?

8. A string can just stand a strain of 20 lbs. weight. If a mass of 5 lbs. be caused to revolve in a horizontal circle at the end of the string, 4 feet long, determine the greatest number of revolutions which it can make in a minute with safety to the string.

9. An engine, whose mass is 9 tons, passes round a curve whose radius is 600 feet with a speed of 30 miles an hour; what force towards the centre is exerted by the rails?

10. A wet umbrella is caused to rotate 14 times in 33 seconds, the handle being vertical. If the diameter of the umbrella is a yard, find the direction and magnitude of the force which will keep a drop of water (mass $\frac{1}{100}$ oz.) attached to the rim.

11. How often per minute must a mass of 10 lbs. revolve horizontally at the end of a string 15 inches long so that the tension of the string may be equal to the weight of a pound?

12. A mass of 3 lbs. is caused to describe a horizontal circle at the end of a string 4 feet long; if the tension of the string be equal to the weight of a pound, how often will the body revolve during the time of a body falling through 200 feet from rest?

13. A mass of 7 lbs. is caused to revolve in a horizontal plane at the end of a string 6 feet long; if the time of a revolution be $1\frac{2}{3}$ seconds, find the tension of the string.

14. A mass of 15 lbs. is suspended by a string from the roof inside a railway carriage; the train at a certain part of its journey is travelling with a speed of 30 miles an hour round a curve of radius 1000 feet; find the tension of the string and its inclination at this time.

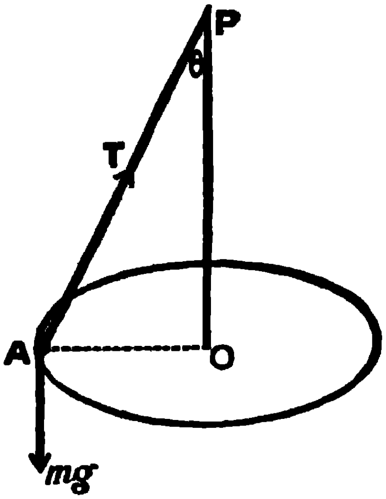
15. A mass of 10 lbs. is caused to rotate in a vertical circle at the end of a string 4 feet long; if the tension of the string vanish at the highest point, find the velocity of the stone in the circle, and the tension of the string, at the instant the stone passes through the lowest point of its path.

16. A mass of 5 lbs. fastened to a string 6 feet long is caused to describe a vertical circle with a uniform velocity of 30 f.-s. Find the tension of the string when the body (1) is at its lowest point; (2) is at its highest point; (3) when the string makes an angle of 60° with the lower vertical radius.

17. A man sent into the 'chains' to call the soundings holds the 'line' at a point 20 feet from the 'lead,' the mass of which is 14 lbs. If the last whirl be described with uniform velocity in 2 seconds, find the tension of the 'line' at the instants the 'lead' passes through its highest and lowest positions.

18. Distinguish between the true and apparent weights of a body resting on the earth's surface. And find how much faster the earth should revolve that a body at the equator may exert no pressure on the surface.

336. If the bob of a pendulum swing in a horizontal circle, the arrangement is known as a **Conical Pendulum.**



Let the string PA, whose length is l, make an angle θ with the vertical, and suppose the bob to move in a circle, whose radius OA is r, with uniform velocity v.

The forces acting on the bob are its weight (mg) downwards, the tension (T) of the string, and the force $\left(\frac{mv^2}{r}\right)$ directed towards the centre which compels it to pursue the circular path.

Resolving the tension vertically and horizontally, and equating the vertical forces, and also the horizontal forces, we have—

$$T\cos\theta = mg \quad \ldots\ldots \quad (1)$$

$$\text{and, } T\sin\theta = \frac{mv^2}{r} \quad \ldots\ldots \quad (2)$$

$$\text{From equation (1)} \frac{m}{T} = \frac{\cos\theta}{g}.$$

$$\text{,, \quad ,, \quad (2)} \frac{m}{T} = \frac{r\sin\theta}{v^2};$$

$$\therefore \frac{\cos\theta}{g} = \frac{r\sin\theta}{v^2} = \frac{l\sin^2\theta}{v^2};$$

$$\therefore \frac{\cos\theta}{\sin^2\theta} = \frac{gl}{v^2}.$$

If t = time of revolution, then $vt = 2\pi r = 2\pi l \sin\theta$;

$$\therefore\ v^2 = \frac{4\pi^2 l^2 \sin^2\theta}{t^2};$$

$$\therefore \frac{\cos\theta}{\sin^2\theta} = \frac{gl \times t^2}{4\pi^2 l^2 \sin^2\theta}.$$

From which equation, $t = 2\pi\sqrt{\dfrac{l\cos\theta}{g}}$.

Comparing this result with the time of oscillation of a simple pendulum, found in Art. 328, we see that a conical pendulum revolves in twice the time of oscillation of a simple pendulum whose length is the vertical projection of the conical pendulum.

MISCELLANEOUS EXAMPLES.—LXIV.

1. From a train moving at the rate of 30 miles an hour a carriage is slipped at the bottom of an incline, and comes to rest at the top of it; find the height of the plane (leaving friction out of account).

2. A shot is fired horizontally with a velocity of 900 f.-s.; find its velocity at the end of 3 seconds.

3. A projectile is fired from a gun laid horizontally from the top of a fort 60 feet above the water; in what time will it reach the ground? Why is it not necessary to give the velocity on leaving the gun?

4. Two bullets are fired simultaneously from the same point in given directions; prove their distance apart increases uniformly with the time.

5. A body is projected at an angle of elevation of 45° with a velocity of $3\sqrt{2}\,.\,g$ from the top of a tower, the height of which is $8g$; how long will it be (1) before it is moving horizontally; (2) before it passes the horizontal plane through the point of projection? At what distance from the foot of the tower will it strike the ground?

6. A shot is fired with a velocity of 1000 f.-s. in a direction inclined at 30° to the horizon; find the greatest height attained, and the horizontal range.

7. Explain the meaning of *mass, momentum, unit of force, density, weight.* What weight is taken as the unit of force?

8. Define 'force,' and state what is the proper measure of force. What is the kinetic unit of force used in England?

9. If the units of space, time, and mass be 88 yards, a minute, and a $\frac{1}{2}$ cwt. of matter respectively, find the measure of momentum of a train of 60 tons, moving with a velocity of 30 miles an hour.

10. On what do you rest your belief in the First Law of Motion?

11. State the Second Law of Motion, and apply it to explain the path of a projectile *in vacuo.*

12. ABC is a triangle. Find a point O within it such that forces, completely represented by OA, OB, OC may be in equilibrium.

13. Enunciate the Third Law of Motion, and explain all the technical words made use of.

14. A bucket is raised from a well in 8 seconds by a mass of 90 lbs. connected with it by a string over a pulley; and in $2\sqrt{10}$ seconds by a mass of 105 lbs.; find depth of well, and mass of the bucket.

15. Two masses m and m_1 are connected by a string over a rough pulley, and motion cannot take place until the tension on one side of the pulley exceeds that on the other side by a constant quantity T_1; find the acceleration.

16. If the weight of a ton be the unit of force, and a minute and a yard those of time and length, what will be the unit of mass?

17. A stone thrown down a rough board inclined at an angle of 30° neither gains nor loses velocity in its descent. What velocity will it gain by falling down the board (which is 20 feet long) when it is inclined at an angle of 60°?

18. Find the Unit of Force, when an *inch*, a *minute*, an *ounce* are the units of space, time, and mass respectively.

19. A mass of 2 lbs. is drawn up a smooth plane inclined at 30° to the horizon by a force acting along the plane. What is the magnitude of the force (in gravitation measure) when, starting from rest at the foot, the body reaches the top in the same time as it would take to slide freely from the top to the bottom under the influence of gravity alone?

20. Three forces P, Q, R, acting at O, are in equilibrium, and any line meets their lines of action in L, M, N; prove that

$$\frac{P}{OL}+\frac{Q}{OM}+\frac{R}{ON}=0.$$

21. Distinguish between *moving* and *impulsive* forces, and state how each kind is measured.

22. A stone A is thrown vertically upwards with a velocity of 96 f.-s.; after 4 seconds from the projection of A another stone B is let fall from the same point. Prove that A will overtake B after 4 seconds more.

23. A body impinges obliquely on a fixed smooth plane; find the velocity after impact, the elasticity being imperfect.

24. Find the velocity and direction of projection of a ball that it may be 64 feet above the ground at a distance of 384 yards, and may strike the ground at a distance of 640 yards.

25. If a constant force P act on a body whose weight is W, the velocity generated in one second is $\frac{P}{W}\cdot g$; find the amount of work done by the force in producing this velocity.

26. A body of mass 65 is moving with a velocity of 91, the units of mass, length, and time being the *lb.*, *foot*, and *second* respectively; express its Momentum and its Kinetic Energy when the units are the gramme (= ·035 oz.), the centimetre (= ·39 in.), and the second.

27. A mass of 5 lbs., after falling freely through 64 feet, commences to raise a mass of 7 lbs. by a string over a smooth fixed pulley; how high will it lift it?

28. The triangle ABC, right-angled at B, has its hypothenuse AC vertical, and D is the middle point of BC. If two equal masses start from rest at the summit A, one falling freely down AC, the other sliding down AB, prove that their C. G. describes AD with uniform acceleration.

29. Two masses, m and m_1 ($m > m_1$), hang over a smooth pulley; prove that the acceleration of their C. G. is $\left(\frac{m - m_1}{m + m_1}\right)^2 g$.

30. A bird whose mass is $\frac{1}{2}$ lb., flying with a velocity of 10 f.-s., is killed instantaneously by a shot whose mass is ·02 oz., and moving with a velocity of 400 f.-s. in a direction at right angles to the bird's; find the tangent which the direction of its motion immediately after makes with that of its flight. Neglect the fact that the bird's mass is increased by the shot.

31. A merchant buys a cargo of tea at Shanghai, where $g = 32·12$, and sells it at St. Petersburg, where $g = 32·20$, by *weight* as indicated in the two places by the same spring-balance which had been tested by standard at Odessa, where $g = 32·16$, and found correct for that latitude. Selling the same apparent weight that he bought, he still holds what his balance calls a ton. Find in standard weight the tonnage of cargo.

32. If the mass of platinum at the Exchequer Office, which weighs 1 lb. in London ($g = 32·19$) be the unit of mass, then

(1) What is the unit of weight in London?

(2) What, in terms of this weight, is the weight of the platinum mass at the Equator ($g = 32·09$)?

33. What becomes of the energy of a projectile when it strikes a target?

34. By what methods may acceleration due to gravity be determined?

35. Three bullets are fired vertically upwards at the end of three successive seconds of time, the first with a velocity of 500 f.-s., the next with a velocity of 600, the third with a velocity of u; determine u, so that all three bullets may pass through a plane parallel to the horizon at the same instant.

36. A train moving at the rate of 30 miles an hour comes to the foot of an incline of 1 in 400. If the steam be turned off, find how far the train will ascend the slope, the resistance to the motion due to friction and the air being equal to the weight of 12 lbs. per ton.

37. A particle starting from A runs down a smooth plane AB inclined at 45° to the horizon. Two seconds before it starts another is projected up the plane from B with a velocity $\frac{5g}{2\sqrt{2}}$, and meets the first one half of the way from B to A. Find when they meet.

38. If three forces be in equilibrium, and the sides of a triangle taken in order represent them in direction, they will also represent them in magnitude; but if five forces be in equilibrium, and the sides of a pentagon taken in order represent them in direction, they will not necessarily represent them in magnitude. Explain this.

39. A body placed in a spring balance at a place where $g=32$ appeared to weigh 14 lbs.; what is its true mass, if the balance was made at a place where $g=32\cdot2$?

40. A uniform beam 9 ft. long rests with one end against a smooth vertical wall, the other on a smooth horizontal plane, and is prevented from slipping by a horizontal force applied at that end, equal to $\frac{2}{3}$ of the weight of the beam, and by a weight equal to the weight of the beam suspended from a certain point of the beam; find the distance of this point from the lower end of the beam if the beam be inclined at an angle of 45° to the horizon.

41. Several forces in different directions act in one plane on a particle; find the conditions for equilibrium.

42. A plank rests over a cliff having $\frac{3}{8}$ of its length projecting over the edge, to which it lies at right angles. An animal whose weight is $\frac{3}{8}$ the plank's crawls along it. When will the equilibrium be disturbed?

43. ABC is a triangle, one side of which, AC, is divided into four equal parts in the points D, E, F; show that if BA, BD, BE, BF, BC represent in magnitude and direction forces acting at the point B, their resultant will be fully represented by $5BE$.

44. ABC is an isosceles triangle, A and B being the equal angles, and from C a perpendicular CD is drawn to AB. If G be a point in CD such that $GD=\frac{1}{3}CD$, show that forces fully represented by GA, GB, GC will be in equilibrium.

45. If three forces act on a rigid body, and are in equilibrium, prove that their lines of action are either all parallel or pass through a point.

46. ABC is a triangle, B and C being fixed points, A moves about in space; show that the direction of the resultant of forces represented in magnitude and direction by AB and AC passes through a fixed point.

47. $ABCD$ is a quadrilateral, and forces are fully represented by AB, BC, AD, DC; show that their resultant passes through the middle point of BD, and is equal to $2AC$.

48. Three forces acting on a particle are in equilibrium. One of the forces is 120, and the others are in the ratio 1 : 2. Find the magnitude of these forces if the angle between their directions be 135°.

49. Forces which are proportional to cos A, cos B, 2 cos C act along the sides of a triangle in the directions BC, CA, BA; show that the direction of their resultant will pass through the intersection of the perpendiculars on the sides from the opposite angles.

50. A uniform lamina of weight W in the form of an equilateral triangle rests with the base on a smooth inclined plane whose inclination is 30°, the plane of the triangle being vertical and at right angles to the inclined plane; find the force to be applied to the vertex in a horizontal direction in order to keep it in equilibrium.

51. The resultant of two forces is $2P$, the magnitude of one is P, and the direction of the other makes an angle of 30° with the direction of the resultant; find the direction of the former and the magnitude of the latter.

52. A sunk fence is formed of two walls, one vertical, and the other inclined at an angle 45° to the vertical; find the position in which a plank will rest against the two walls, the surfaces being smooth.

53. Two weights P and $2P$ are in equilibrium on a double inclined plane, connected by a string passing over a smooth peg vertically over the highest point of the plane; the angle at the vertex of the plane is 90°; find the condition that the strings may be equally inclined to the two planes.

54. Three equal forces OA, OB, OC act at O in the same plane. If L be the point of intersection of the perpendiculars from A, B, C on the opposite sides of the triangle ABC, prove that OL will represent the resultant of the forces.

55. $ABCDEF$ is a regular hexagon inscribed in a circle. Forces P, nP act on a particle at A, in the directions AD, FA respectively; find n, in order that their resultant may act along the tangent to the circle at A, and determine the magnitude of the resultant.

56. A heavy uniform rod AB (weight $= W$), moveable about a horizontal axis at A, has fastened to it at B a string, which, after passing over a small pulley at C in the plane of the rod's motion, and in the same horizontal line as A, supports a weight W. If $AB = AC$, and B be above AC, find the position of equilibrium.

57. AB and CD are any two equal and parallel chords of a given circle. P is a point on the circumference between A and B. Show that if forces completely represented by PA, PB, PC, PD act at P, their resultant is of constant magnitude.

58. Show how to find the centre of any number of parallel forces acting on a rigid body. Apply proposition to define the C. G. of a body.

59. Give examples of the three classes of levers. Find the condition for equilibrium, and show in which of the classes Mechanical Advantage is lost or gained.

60. A heavy uniform board in the form of an isosceles triangle, each of whose equal sides is three times the base, is suspended from one extremity of the base. What weight must be suspended from the other extremity of the base in order that the sides through the point of suspension may be equally inclined to the horizon?

61. Prove that when an inclined plane is *rough* the mechanical advantage in dragging a body up by a force parallel to the plane is not the same as that in letting it down.

62. A ball whose mass is 6 lbs., resting on a smooth table, is connected by a string 10 feet long with a mass of 2 lbs. just over the edge at a height of 4 feet; in what time will the 6 lbs. reach the ground, at what distance from the other body, and with what momentum?

Two weights, P and Q, are suspended by fine silk threads from the extremities of the base of a lamina in the form of an isosceles triangle ABC, whose weight is W, and which is free to move about a horizontal axis through the vertex A; find the inclination of the base to the horizon, h being the altitude of the triangle, and $2a$ the length of its base.

63. A body is kept at rest by the forces P, Q, R, S acting along the sides of the quadrilateral $ABCD$; prove that—

$$P \times R : Q \times S :: AB \times CD : AD \times BC.$$

64. A body is acted on by forces completely represented by the sides BA, BC, and the diagonals CA, BD of a square. Find the magnitude of the force along DC, in order that the whole system of forces may reduce to a single resultant passing through A; and when this is the case, find the magnitude and direction of this resultant.

65. Two small rings slide on the arc of a vertical circle. A string passing through them has three equal weights attached to it at each end, and one between the rings. Find the position of equilibrium.

66. The points A, B, C and D, E, F are taken in two parallel straight lines, AC and DF respectively, and five forces represented in magnitude and direction by AD, DB, BE, EC, CF; prove that their resultant is completely represented by AF.

67. $ABCD$ is a square; E, F are the middle points of CD, DA respectively. Forces are represented in magnitude and direction by AB, EF; find the magnitude of the force which must act along BD so that the three forces may have a resultant which passes through C.

68. A heavy uniform rod *AB* $4\sqrt{6}$ feet long is in equilibrium at an angle of 15° to the vertical, with its lower extremity *A* resting on a smooth wall inclined at an angle of 60° to the horizon, and a point in it, *C*, pressing against a smooth horizontal peg; prove that the distance of *C* from the wall is $(\sqrt{3}-1)$ feet.

69. *ABCDEF* is a regular hexagon; forces *P*, *Q* act at *A* in directions *AC*, *AF* respectively. Find the ratio of *Q* to *P* in order that their resultant may act along *AD*, and find magnitude of the resultant.

70. A straight line is drawn through *A*, one of the angles of a triangle *ABC*, so as not to cut the opposite side; show that the distance of the C. G. of the triangle from this line is $\frac{1}{3}$ of the sum of the distances of *B* and *C* from it.

71. A piece of uniform board, in the form of two isosceles triangles on opposite sides of a common base, will just rest in a vertical plane with a side in contact with a horizontal plane surface. If $2a$, 2β be the vertical angles of the triangles, prove that

$$\cot\beta = \cot a + 3\tan a.$$

72. If *G* be the centre of three equal parallel forces acting at *A*, *B*, *C*, *O* being any other point, the resultant of three forces represented in magnitude and lines of action by *OA*, *OB*, *OC* is represented by 3 *OG* along *OG*.

73. *ABC* is a triangle, *D*, *E*, *F* the middle points of *BC*, *CA*, *AB* respectively; prove that the forces represented by *AD*, *BE*, *CF* maintain equilibrium.

74. Three parallel forces act at the vertices, *A*, *B*, *C*, of a triangle in a direction at right angles to the plane of the triangle, and their resultant acts through the C. G. of the triangle; prove that the forces are equal.

75. If at a point on the circumference of a circle two unequal forces act, represented in magnitude and direction by chords of the circle, find the condition that the resultant may pass through the centre.

76. A triangular board is kept in equilibrium by forces acting upon its three sides (one upon each side) in directions perpendicular to them; show that these forces are proportional to the sides on which they act.

77. One of two forces is given in direction but not in magnitude, and the resultant of the two is given in both magnitude and direction; find the least possible magnitude and corresponding direction of the second force.

78. A right-angled triangular board of sides $2\frac{1}{2}$, 6, $6\frac{1}{2}$ feet is in equilibrium in a vertical plane, with its side of 6 feet horizontal and resting on a smooth peg, and with its vertex opposite that side fastened to the lower end of a string attached to a fixed point. Prove that this string must be vertical; and find the least possible horizontal distance between the peg and the right angle.

79. Part of a heavy uniform cord lies on a smooth inclined plane, and the rest hangs vertically from the top of the plane; show that the ends of the cord are in the same horizontal plane.

80. If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium; prove this proposition.

If for one of the forces another were substituted, not acting at the point, but having the same direction and double the magnitude, what would be the resultant of the system?

81. If L, M, N be the vertices and P, Q, R the position of the C. G.'s of the three equilateral triangles described externally on the sides of a triangle ABC, then having given that the C. G.'s of the triangles ABC, LMN coincide, prove that the C. G. of the triangle PQR is at the same point.

82. A weightless rod is moveable in a plane round a pivot at its middle point; two strings attached to its extremities intersect on a fixed straight line in the plane, and are pulled with equal and constant forces. When the rod is in equilibrium in a given position, where must their point of intersection be? Find also the position of the rod in which the pressure on the pivot is least.

83. Obtain from the Laws of Motion a Definition, and also a Measure of Force.

84. $ABCD$, $AB'C'D'$ are two parallelograms with one point common; prove that forces parallel and proportional to BB', $C'C$, DD', and all acting at the same point will be in equilibrium.

85. Forces P, Q, R . . . X, Y acting at a point are in equilibrium; the magnitudes of all but X, and the directions of all but Y are known. Show by a geometrical construction how the unknown quantities may be determined.

86. A uniform rod AC of weight W can turn in a vertical plane about a hinge at A. A string passing round a fixed pulley B has one end attached to C, and the other to a weight P which hangs freely. If AB be horizontal and equal to AC, find the ratio of P to W in order that ABC may be an equilateral triangle. Find also the strain on the hinge.

87. What is meant by the Modulus of a machine?

88. A piece of uniform wire bent into the form of three sides of a square is hung up by one angle: find the position of equilibrium.

89. If a and b are the diameters of the non-concentric spheres forming a shell, and d is the distance between their centres, find the distance of the C. G. of the shell from the centre of the larger sphere whose diameter is a. Assume that the vol. of a sphere $\propto$ (radius)^3.

90. A triangular slab of uniform thickness is supported at its three angles; whatever be the form of the triangle, show that the pressures on the props are equal.

91. If a weight be placed anywhere on a triangle resting on props at its angles, show that the pressures on these props are proportional to the areas of the opposite triangles formed by drawing lines from the position of the weight to the angles.

92. The sides of a triangle are 3, 4, 5, and its inscribed circle is removed; find the C. G. of the remainder.

93. Through what chord of a circle drawn through its lowest point must a body fall so as to acquire a velocity equal to $\frac{1}{n}$th of the velocity due to falling down the vertical diameter?

94. Find that diameter down which a body when falling will describe the lower half in the same time as another body will take to fall down the vertical diameter.

95. If W be the weight of an empty omnibus, h the height of its C. G., h_1 the height of the roof, W_1 the greatest weight which may be placed upon the roof, so that the 'bus shall not upset when tilted through an angle θ, and the distance between the wheels be $2x$, find W_1.

96. A body whose weight is P having fallen through h feet begins to pull up a heavier body whose weight is Q by means of a string passing over a smooth pulley. To what height does it lift it?

97. If the distances of the points of application of n equal like parallel forces from a plane through the point of application of the resultant form a descending A. P. of which the first term is a, and the common difference is b, show that $\frac{a}{b} = \frac{n-1}{2}$.

98. An engine drives the shaft of a train of machinery by a strap passing round a wheel 6 feet in diameter centred on the shaft. If the difference in the tensions of the strap on either side of the wheel be 25 lbs. weight, and if the shaft make 7 revolutions per second, find the horse-power of the engine. Take $\pi = \frac{22}{7}$.

99. If in a false balance, the length of which is l, a body weigh p at one end and q at the other, express the lengths of the arms in terms of p and q and l.

100. The beam of a false balance being uniform, show that the arms are respectively proportional to the differences between the true and the false weights of a body.

101. Parallel forces act at the angles ABC of a triangle and are proportional to $a \cos A$, $b \cos B$, $c \cos C$, respectively; show that their centre coincides with the centre of the circumscribing circle.

102. Like parallel forces, 3, 5, 7, 5 act at A, B, C, D, the angular points of a quadrilateral $ABCD$ taken in order; show that their centre will remain unchanged if, instead of these forces, we have parallel forces P, $10-P$, $4+P$, $6-P$ acting at the middle points of AB, BC, CD, DA, where P may have any value.

103. Find the centre of equal and like parallel forces acting at seven of the angular points of a cube.

104. Parallel forces P, Q, R, act at the angular points A, B, C of a triangle, and their centre is at O; show that $P : Q : R = \triangle BOC : \triangle COA : \triangle AOB$.

105. The sides of a heavy triangle are 3, 4, 5 respectively; if it be suspended from the centre of the inscribed circle, show that it will rest with the shortest side horizontal.

106. The altitude of a solid cone is h, and the diameter of the base is b; a string is fastened to the vertex and to a point in the rim of the base, and is then hung over a smooth peg; if the cone rest with its axis horizontal, show that the length of the string $= \sqrt{h^2+b^2}$.

107. A ship is moving with a velocity u; a cannon is fired at an elevation α; the velocity produced by the charge of powder is v when the ship is at rest. Find the range, supposing that the ship and the shot are moving in the same vertical plane.

108. A body projected along a horizontal surface with a velocity of 20 f.-s. is brought to rest in 400 yards; find the coefficient of friction and the time of motion.

109. The numerical measure of a certain force is 56 in the 'foot-pound-second' system of units; find its measure in the 'yard-cwt.-minute' system of units.

110. A train travelling 50 miles an hour comes into collision with a fixed object. From what height must a person fall in order to receive a blow equal to that sustained by him in the collision? ($g = 32{\cdot}2$.)

111. A shot is fired vertically from a gun, the barrel of which is 30 inches long. It rises 2640 feet. Compare the acceleration caused by the discharge with that caused by gravity.

112. With what force will a 10 lbs. mass falling 100 feet strike a fixed surface if its motion be destroyed in $\frac{1}{2000}$ second? ($g=32 \cdot 2$.)

113. If the ball in Ex. 112 be flattened $\frac{1}{30}$ inch, and then recovering its form rebound, find (1) the retardation, (2) the time taken to bring the ball to rest, (3) the mean pressure between the surfaces.

114. A uniform rod is freely moveable about a hinge at one end, and is supported by a string attached to the other; show that if the direction of the reaction at the hinge be equally inclined to the rod and the vertical, then the direction of the string is at right angles to that of the action at the hinge.

115. A right circular cone whose vertical angle $= 2\alpha$ stands on a rough plane which is gradually tilted up; show that if $4 \tan \alpha = \tan \mu$, where μ is the angle of friction, it will be uncertain whether the cone will first slip or upset.

116. Two parties are pulling against each other in a 'tug-of-war,' and neither overcomes the other; what are the forces in equilibrium?

What forces become unequal when one party is prevailing?

117. A uniform ladder, 16 feet long, is being lifted up about one end by a force applied at right angles to the ladder at 6 ft. distance from that end; show that when the ladder makes an angle of 30° in the ground, the reaction at that end is horizontal.

118. The horizontal roadway of a bridge is 30 ft. long, and weighs 6 tons; it rests on similar supports at its ends. What pressure is borne by each support when a carriage weighing 2 tons is one-third of the way across the bridge?

119. A uniform heavy rod AB is fastened at A to a vertical wall, and is kept in a horizontal position by means of a weightless rod CD inclined at an angle of 45° to it; one end D is fastened to the wall below A, and the other C to AB, midway between A and the middle point; show that the reactions at A and C are respectively $\sqrt{5}W$ and $2\sqrt{2}W$, where W is the weight of the rod.

120. Three equal forces each of 1 lb. weight act along three sides of a square taken in order; find the force which will balance them.

121. AB, AC, are two straight lines inclined at an angle 2θ; R is the resultant of forces P and Q acting along AB, AC respectively; R_1 is the resultant of forces P and Q acting along AC, AB respectively; find the resultant of R and R_1.

122. Forces 1, 2, 3, 4 act along the sides AB, BC, CD, DA of a square; find their resultant.

123. If through a fixed point there be drawn three straight lines representing three forces in equilibrium, any one of them will if produced bisect the line joining the extremities of the other two.

124. AD, BE, CF are the perpendiculars from the angles of a triangle on the opposite sides; show that forces acting at a point parallel and proportional to AD, BE, CF, FA, DB, EC are in equilibrium.

125. A smooth uniform rod of length $2l$ rests over a peg and has one end in contact with a smooth inclined plane; show that if α be the inclination of the plane and d the distance of the peg from it, the rod will make an angle θ with the vertical given by the equation,—

$$l \sin \theta \cos^2 (\alpha - \theta) = d \sin \alpha.$$

126. Forces 1, 6, 3, 4, 5, 2 act at a point in a plane making equal angles with each other; find their resultant.

127. Find the resultant of any number of forces acting in one plane on a body, and deduce the necessary and sufficient conditions of equilibrium.

128. The arc of a semicircle $APQRSB$ is divided into five equal arcs in the points P, Q, R, S; show that the resultant of five equal forces acting along AP, AQ, AR, AS, AB respectively is inclined at an angle of 36° to the diameter AB.

129. $ABCDEF$ is a regular hexagon; find the magnitude of the resultant of four forces $3P$, $4P$, $6P$, $2P$ which act in the directions AB, AC, AD, AE.

130. Six forces 6, $\sqrt{2}$, 4, 5, 6, $\sqrt{2}$ act at a point, and include angles of 45°, 75°, 60°, 30°, 15° respectively; find the magnitude of the force which will keep them in equilibrium.

131. A rectangular lamina $ABCD$, having $AB = 4$ ft., $BC = 3$ ft., and weighing 30 lbs., movable about a horizontal axis at A, is kept in equilibrium with AD vertical by the tension of a string acting along DB; find the pressure on A.

132. $ABCD$ is a quadrilateral divided into equal triangles by the diagonal BD; prove that the system of forces fully represented by DA, AB, DC, CB, $2BD$ is in equilibrium.

133. The sides AB, AC of a triangle ABC are bisected at D and E; prove that the resultant of the forces represented in magnitude, and line of action by BE and DC is parallel to BC and represented in magnitude by $\frac{3}{2}BC$.

134. A particle is projected at an angle of 60° to the horizon from a given point on an inclined plane of 30°; if R_1, R_2 be the ranges when the particle is projected up and down respectively, prove that $R_2 = 2R_1$.

135. A, B, C are three tacks in a vertical plane; BC is horizontal, A is below and to the left of BC, so that AB is inclined at an angle 60° to the horizon; a string hanging over the tacks supports equal masses of 3 lbs. at its ends. Find the pressure on each tack.

136. A merchant using a spring balance buys goods at x_1 shillings per cwt. at a place where the measure of the acceleration due to gravity is g_1, and sells at x_2 shillings per cwt. at a place where the measure is g_2; find his gain per cent., and show that it is independent of the value of g at the place where the balance was graduated.

137. In example 135, if AB be inclined at an angle θ, and equal weights W hang at the ends of the string, find the pressures exerted.

138. A uniform ladder 10 feet long rests with one end against a smooth vertical wall, and the other on the ground, the coefficient of friction being $\frac{1}{3}$; find how high a man (whose weight is four times that of the ladder) may rise before it begins to slip, the foot of the ladder being 6 ft. from the wall.

139. Forces P, Q, R act along the sides of a triangle ABC; show that their resultant will act along the line joining the centres of the inscribed and circumscribed circles of the triangle if

$$P : Q : R = \cos B - \cos C : \cos C - \cos A : \cos A - \cos B.$$

140. It is found that the greatest distance that the foot of a ladder 65 feet long can be placed from the foot of a smooth vertical wall, against which it rests, is 33 feet. What proportion of the maximum friction possible is called into play when the foot of the ladder is 25 feet from the wall?

141. Two scale-pans, each of mass m, are connected as in Atwood's machine, and in them are placed two bodies whose masses are m_1 and m_2 respectively. Show that the pressures on the pans are—

$$\frac{2m_1(m_2+m)}{m_1+m_2+2m}g \text{ and } \frac{2m_2(m_1+m)}{m_1+m_2+2m}g \text{ respectively.}$$

142. A beam, divided at its C. G. into two parts a and b, lies wholly within a hemispherical bowl. If r be the radius of the sphere, and θ, θ_1 be the angles at the centre subtended by a and b respectively, find the values of θ and θ_1.

143. A train is crossing a viaduct over a river with a speed of 60 miles an hour, the height of the carriage window above the water being 144 feet; at what distance from the end must a passenger drop a stone so as just to strike the water's edge?

144. A projectile whose mass is $\frac{3}{4}$ ton leaves the bore of a gun 30 feet long with a muzzle velocity of 1600 f.-s.; find the steady push exerted by the powder gas.

145. A projectile whose mass is 1000 lbs. strikes a target with a velocity of 1500 f.-s.; find the penetration if the resistance of the plate be equal to the weight of 8000 tons.

146. If a yard be the unit of length, a pound the unit of mass, and a minute be the unit of time, show that the new unit of work is $\frac{1}{400}$ foot-poundals.

147. A bullet whose mass is 1 oz. strikes a plank 3 inches thick with a velocity of 1400 f.-s., and emerges on the other side with a velocity of 1200; through how many similar planks will the bullet make its way?

148. Two particles leave the summit of an inclined plane, one sliding down the slope, the other falling vertically; show that the line joining them at any instant is perpendicular to the length of the plane.

149. A small shell explodes in all directions into equal fragments; if the velocity of the particle which rises vertically will carry it to a height of 81 feet, find the area of danger on the plane.

150. A mass of 3 lbs. is placed on the slope of a right-angled wedge whose inclination is 60°; through what space must the wedge be moved along a table in a second so that the body shall be at rest relatively to the wedge, and find the pressure exerted by the body on the wedge?

151. A driving belt in a machine is moving with a velocity of 30 f.-s.; find the H.P. of the engine if the difference of the tensions on each side of a small pulley be equal to the weight of 40 lbs.

152. The area of a piston is 2000 square inches, and it makes twenty double strokes per minute, the length of the stroke being $4\frac{1}{2}$ feet. If the pressure of steam be equal to the weight of 25 lbs. per square inch, and the efficiency of the machine be $\frac{4}{5}$, how many gallons of water can the engine raise from a depth of 200 feet in a working day of 10 hours?

NOTE.—A gallon of water = 10 lbs.

153. A train whose mass is 200 tons is drawn by a locomotive of 250 H.P.; if the friction and the pressure of the air offer a resistance of $12\frac{1}{2}$ lbs. weight per ton, what is the greatest speed which the train can have on the level?

154. Explain why the pressure of a jet of water from a fire-engine varies as the square of the velocity.

155. A jet of water whose section is 3 square inches strikes an inelastic wall with a velocity of 90 f.-s.; find the pressure on the wall. A cubic foot of water $= 62\frac{1}{2}$ lbs.

156. If in Ex. 155 the velocity of recoil be 10 f.-s., find the pressure on the wall.

157. If 400 gallons of water fall from a height of 40 feet every five minutes, what is the pressure on the ground below, assuming that the water runs off?

158. If in a storm which raged for two hours the steady rainfall was $1\frac{1}{2}$ inches, find the pressure on the ground during the storm, on the supposition that the rain fell through 1000 feet.

159. Expose the fallacy in the following reasoning: 'If a particle be projected vertically downward from a given height h, with an initial velocity u, its velocity on striking the ground must be $u + \sqrt{2gh}$.'

160. A skater, moving with a uniform velocity 10 f.-s., throws a ball horizontally in such a manner that its direction is at right angles to his original path, and its velocity is 50 f.-s.; find the impulse on the ball, and his own path immediately after, on the supposition that his mass is 500 times the mass of the ball.

161. A body projected vertically upwards with a velocity of 20 miles a minute passes at the end of each second through an obstruction which at once reduces the striking velocity by 160 f.-s.; how high will the body rise?

162. A body is projected with a velocity u in a direction at right angles to that of the sun, and in the same vertical plane with it. Show that the retardation of the body's shadow on the horizontal plane through the point of projection is $g \cot \alpha$, and that the shadow will be at its greatest distance from its initial position after a time $u \sec \alpha / g$, α being the sun's altitude.

163. Find the shortest time (from rest to rest) in which a chain, capable of bearing a safe load of P tons, can lift a mass of W tons out of a hold h feet deep.

164. Prove that the greatest load which can be lowered gently into the hold of a vessel h feet deep by a chain capable of bearing a safe load of W tons in t seconds from rest to rest is

$$W\left(1 - \frac{h}{\frac{1}{2}gt^2}\right) \text{ tons}$$

APPENDIX I.

TYPICAL EXAMPLES WORKED OUT.

Example i. Find the work done in raising a cubical block of stone, whose weight $= W$, and whose edge is a,

(1) Upright on one edge.
(2) Upright on one angle.

(1) The length of a diagonal of a side $= a\sqrt{2}$.

When lying flat, the C. G. is $\frac{a}{2}$ above the ground.

When upright on one edge, the C. G. is $\frac{a\sqrt{2}}{2}$ above the ground;

$\therefore$ C. G. is raised $\frac{a\sqrt{2}}{2} - \frac{a}{2} = \frac{a}{2}(\sqrt{2}-1)$.

But, Work done = whole weight × distance C.G. is raised (Art. 308)

$$= W\frac{a}{2}(\sqrt{2}-1).$$

(2) The length of a diagonal of the cube $= a\sqrt{3}$.

When upright on one angle, the C. G. is $\frac{a\sqrt{3}}{2}$ above the ground;

$\therefore$ Work done in raising it from its original position to its position upright on one angle

$$= W\frac{a}{2}(\sqrt{3}-1).$$

Example ii. An oarsman pulls a pair of sculls, each with a force P. If the loom be $\frac{1}{m}$th of the length of the oar from the rowlock, find the force urging the boat on her way.

Consider one of the oars. Let the length of the oar be l.

Let R = pressure at the rowlock.

Take moments about the end of the blade (supposed to be momentarily in contact with a stone beneath the water, and therefore fixed).

$$\text{Then, } R\left(l-\frac{l}{m}\right)=Pl;$$

$$\therefore\ R=\frac{Pm}{m-1};$$

$$\therefore\ \text{Total force at the two rowlocks} =\frac{2mP}{m-1}.$$

Now, since the man must be considered as part of the mass moving with the boat, and since by the Third Law of Motion his hands are pulled towards the stern by a force $2P$;

$$\therefore\ \text{Force urging boat on her way}=\frac{2\,mP}{m-1}-2P=\frac{2\ P}{m-1}.$$

Example iii. Find the pressure per acre due to a steady fall of rain of two inches in 12 hours, assuming that the water falls from a height of 900 feet, and does not rebound.

Mass of water which falls on each square foot $=\frac{1}{6}\times\frac{125}{2}$ lbs.

Velocity on reaching the ground $=\sqrt{64\times 900}=240$ f.-s.;

$\therefore$ Momentum destroyed *per second* by reaction from ground

$$=\frac{125}{12}\times\frac{240}{12\times 60\times 60}\ \text{units of momentum};$$

$$\therefore\ \text{Pressure on 1 sq. ft.}=\frac{125\times 240}{144\times 3600}\ \text{poundals.}$$

$$\therefore\ \text{Pressure on 1 acre}=\frac{125\times 240\times 4840\times 9}{144\times 3600\times 32}\ \text{lbs. weight.}$$

This reduces to 78·77... lbs. weight.

Example iv. Water is thrown by a fire-engine in a continuous stream, and strikes a wall at right angles with a velocity of 40 f.-s. If $e=\frac{1}{5}$, and the section of the hose be 3 square inches, find the pressure on the wall, and express its value in poundals.

Mass of water reaching wall *per second* $=\frac{40\times 12\times 3}{1728}\times\frac{125}{2}$ lbs.

$$\text{Change of velocity}=40+\frac{40}{5}=48\ \text{f.-s.}$$

$$\text{Now, } I=mv \text{ (by Art. 136)},$$

$$=\frac{40\times 12\times 3}{1728}\times\frac{125}{2}\times\frac{48}{1},$$

$$=2500\ \text{poundals.}$$

Example v. A mass of 32 lbs. hanging over the edge of a rough table is connected with a mass of 40 lbs. placed on the table. Find the heat generated while the 32 lbs. descends 60 feet and has its velocity increased from 0 to $8\sqrt{47}/3$ f.-s.

NOTE.—Assume that 772 ft.-lbs. = 1 B. T. U. (Art. 319).

The K. E. of the system at starting + Work done by gravity on the system = final K. E. of the system + Work done against the friction;

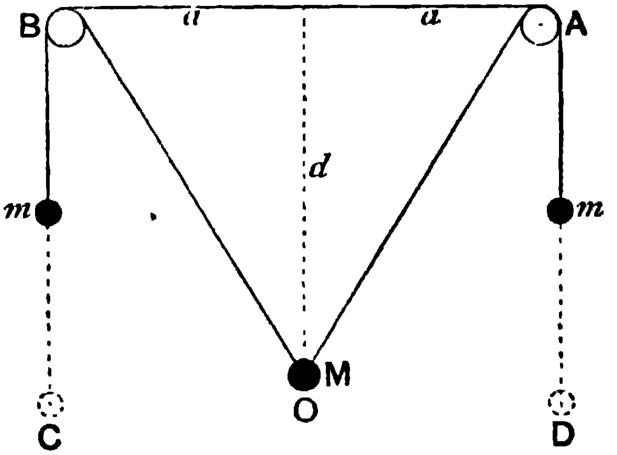
$$\therefore\ \tfrac{1}{2}(40+32)\cdot 0+32g\cdot 60=\tfrac{1}{2}(40+32)\left(\frac{8}{3}\sqrt{47}\right)^2+\text{Heat produced};$$

$$\therefore\ 0+1920\,g=\frac{36\times 64\times 47}{9}+\text{Heat produced.}$$

From this equation, the Heat = 1544 ft.-lbs.,

$$=\frac{1544}{772}=2\text{ B. T. U.}$$

Example vi. Two equal masses (each $=m$) are made fast to the ends of a light inextensible string which is then placed over two smooth pegs in the same horizontal line. If a mass M is then attached to the string midway between the pegs and be allowed to fall, how far will M descend before returning to a position of equilibrium?



Let M descend a distance d to O, and in consequence the masses at the ends of the string ascend to the positions indicated in the figure from their original positions at C and D.

Now the length of string which slips over each peg must $= OB-a$;

$$\therefore = \sqrt{d^2+a^2}-a.$$

The loss of energy of M in its lowest position at O must be equal to the gain of energy of m and m at their increased heights;

$$\therefore\ Mgd=2mg\left(\sqrt{d^2+a^2}-a\right).$$

From which equation the value of d may be determined.

Example vii. A train of 200 tons is drawn by a locomotive of 300 H.P. along a level railway. The resistance due to friction being taken as 15 lbs. weight per ton, required (1) the *greatest speed* which the engine can maintain, and (2) the *acceleration* of the train at the instant its velocity is 20 miles per hour, the engine being supposed to be at full power at that instant.

(1) *To find the maximum speed*—

The engine yields $(300 \times 550g)$ ft.-pdls. per second.

If v = greatest speed in miles per hour;

$\therefore \frac{22v}{15}$ = greatest speed in feet per second;

$\therefore$ Work done against Friction per second $= 200 \times 15g \times \frac{22v}{15}$ ft.-pdls.;

$$\therefore 200 \times 15g \times \frac{22v}{15} = 300 \times 550g.$$

From this equation v is found to be $37\frac{1}{2}$;

$\therefore$ Maximum speed is $37\frac{1}{2}$ miles per hour.

(2) *To find the acceleration when the speed is 20 miles per hour*—

At any instant, the Work done against Friction per second + Work done by the pull of the engine per second = Work yielded by the engine per second.

Let a = acceleration at the instant in question;

Then, since $F = ma$, and 20 m. h. $= \frac{88}{3}$ f.-s., we have

$$(200 \times 15g)\frac{88}{3} + (200 \times 2240a)\frac{88}{3} = 300 \times 550g,$$

$$\text{or, } (200 \times 15g + 200 \times 2240a)\frac{88}{3} = 300 \times 550g.$$

From which equation, $a = \frac{3}{16}$ f.-s.-s.

Example viii. A mass of 1 lb. moving horizontally on a smooth table with a velocity of $64\frac{1}{4}$ f.-s. drives a nail of mass $\frac{1}{16}$ oz. into a fixed block of wood. If the nail penetrate $\frac{1}{4}$ inch, find the mean resistance, there being no recoil on striking.

To find the velocity with which the nail begins to penetrate.

By Law iii. Momentum after the blow = Momentum before the blow.

Let v = Vel. required;

$$\therefore \left(1 + \frac{1}{256}\right)v = 1 \times 64\frac{1}{4};$$

$$\therefore v = 64 \text{ f.-s.}$$

Let R = mean resistance, measured in poundals.

Then, Work done against this resistance = Loss of K. E.;

$$\therefore\ R\cdot\frac{1}{48}=\frac{1}{2}\left(1+\frac{1}{256}\right)\cdot 64^2.$$

From this equation, $R=98688$ pdls.;

$$\therefore\ R=3084 \text{ lbs. wt.}$$

Example ix. If the block be a mass of $8\frac{255}{256}$ lbs. and be quite free to move, how far does the nail penetrate, the resistance remaining the same?

As above, the nail begins to move with a Vel. = 64 f.-s.

Then, Work done against the resistance = Loss of K. E.

Let x = distance nail enters the wood in inches;

$$\therefore\ 3084g\times\frac{x}{12}=\frac{1}{2}\left(1\frac{1}{256}\right)\cdot 64^2-\left(1\frac{1}{256}+8\frac{255}{256}\right)v^2 \quad \ldots\ldots\ldots \quad (1.)$$

where v = velocity of the system after the nail has been struck, and v can be found from the fact that

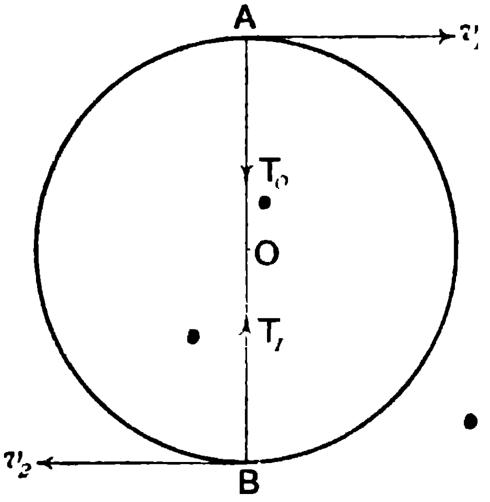
Momentum after the blow = Momentum before the blow;

$$\therefore\ \left(1\frac{1}{256}+8\frac{255}{256}\right)v=1\frac{1}{256}\times 64 \quad \ldots\ldots\ldots\ldots \quad (2.)$$

From (2), we find that $v=\frac{257}{40}$ f.-s.

Substituting in (1), we find that $x=0\cdot2$ inches.

Example x. A mass m is caused to rotate round a fixed point in a vertical circle. Find the least velocity which the body must have at its highest and lowest points respectively, so that the tension of the string may become zero as the body passes through the highest point of the circle. Find also the least possible strength of the string.



Let v_1 = velocity at A. V_2 = velocity at B.
Then, $T_0 + mg$ = Force along AO;

$$\therefore\ T_0 + mg = \frac{mv_1^2}{r},\ \text{where } r = \text{radius of the circle.}$$

But $T_0 = 0$, by hypothesis;

$$\therefore\ v_1 = \sqrt{rg}.$$

When the body passes through B, it has lost *Potential* Energy $= mg \cdot 2r$, but it has gained an equivalent amount of K. E.;

$\therefore$ K. E. at B = K. E. at A + P. E. at A;

$$\therefore\ \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}mv_1^2 + mg \cdot 2r;$$

$$\therefore\ v_2^2 = rg + 4rg = 5rg;$$

$$\therefore\ v_2 = \sqrt{5rg}.$$

When m is at B_1 the force acting along $BO = T_1 - mg$;

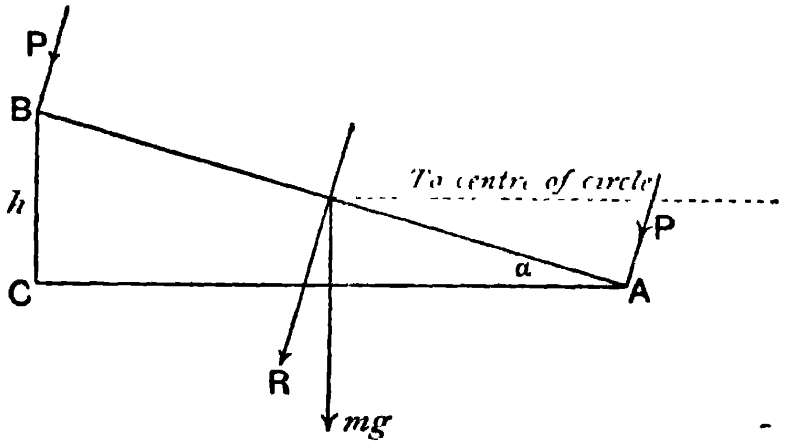
$$\therefore\ T_1 - mg = \frac{mv_2^2}{r};$$

$$\therefore\ T_1 = mg + 5mg;$$

$$= 6mg.$$

Hence the string must be strong enough to support a mass of $6m$. hanging at rest.

Example xi. A train travels on a railway curve of radius r with a uniform velocity v. Find the height of the outer rail above the inner, in order that there may be no wearing of the flanges of the wheels, the distance between the rails being x.



Let AC be the horizontal line through A; AB the 'way' $= x$.
Then BC is the height of the rail B above the rail A; let $BC = h$.
Now all pressure on the flanges is evidently removed when the pressure on the rails is perpendicular to the 'way.'

Let P = Pressure on each rail, then $2P = R$ the resultant pressure on the two rails. The Force (R_1) on the train is *equal* and *opposite* to R.

$\therefore$ the *vertical* component of R_1 must be mg, where mg is the weight of the train of mass m.

and the *horizontal* component of R_1 must be the force causing the train to move on the curve;

$$\therefore\ R_1 \sin \alpha = \frac{mv^2}{r} \ldots\ldots\ldots\ldots\ldots \text{(1.)} \quad \text{(Art. 335.)}$$

$$\text{and, } R_1 \cos \alpha = mg \ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

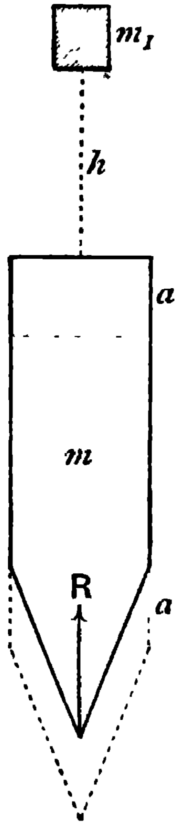
Divide (1) by (2) $\therefore \tan \alpha = \dfrac{v^2}{rg}$.

$$\text{But, } \tan \alpha = \frac{h}{\sqrt{x^2 - h^2}};$$

$$\therefore \frac{h}{\sqrt{x^2 - h^2}} = \frac{v^2}{rg}.$$

And h can be found from this equation.

Example xii. A pile of mass m lbs. is driven vertically a feet into the ground by a mass of m_1 lbs. falling on it from rest through a distance h feet. If there be no recoil, find the mean resistance of the ground.



The velocity of m_1 on reaching the pile $= \sqrt{2gh}$.

Let v = velocity with which $(m + m_1)$ begins to move after the blow.

By Law iii. $(m + m_1)\, v = m_1 \sqrt{2gh}$;

$$\frac{}{m + m_1} \cdot \sqrt{2gh}.$$

Let R = mean resistance offered by the ground.

The forces acting on $(m + m_1)$ are evidently R upwards, and the weight of $(m + m_1)$ downwards.

Hence, the K. E. of $(m + m_1)$ measures the work done against the *resultant* of these two forces acting through the distance a feet;

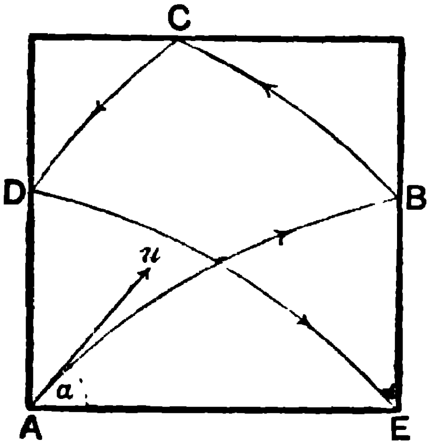
$$\therefore \{R - (m + m_1)g\}a = \tfrac{1}{2}(m + m_1)v^2$$

$$= \tfrac{1}{2}(m + m_1) \cdot \frac{m_1^2 \cdot 2gh}{(m + m_1)^2}.$$

From this equation we shall find that

$$R = \frac{m_1^2 h + (m + m_1)^2 a}{(m + m_1)a} \text{ lbs. wt.}$$

Example xiii. A smooth ball is projected from one of the base angles of a square with a velocity u at an angle of elevation α. Find the condition that it may strike three sides in succession and reach the other base angle, the coefficient of elasticity being e, and the side of the square being x.



Resolve u into $u \sin \alpha$ vertically, and $u \cos \alpha$ horizontally.

Then we note that:

(1) Velocity of recoil $= e \times$ Velocity of approach (Art. 272); and

(2) The vertical velocity at B and D, and the horizontal velocity at C are not affected by impact, the ball being smooth.

Now the body is moving horizontally and also vertically during the *whole time of motion* (T).

Considering the *horizontal* motion only, we see that—

$T =$ Time from A to B + Time from B to D (*viâ* C) + Time from D to E.

$$= \frac{x}{u \cos \alpha} + \frac{x}{eu \cos \alpha} + \frac{x}{e^2 u \cos \alpha};$$

$$= \frac{x}{u \cos \alpha}\left(1 + \frac{1}{e} + \frac{1}{e^2}\right) = \frac{x(1+e+e^2)}{e^2 u \cos \alpha} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.)$$

Considering the *vertical* motion only, we see that—

$T =$ Time from A to C (*viâ* B) + Time from C to E (*viâ* D).

Let $t_1 =$ time from A to C, and $t_1 =$ time from C to E;

Using $s = ut + \frac{1}{2} a t^2$;

$\therefore\ x = u \sin \alpha \cdot t_1 - \frac{1}{2} g t_1^2$ which gives t_1;

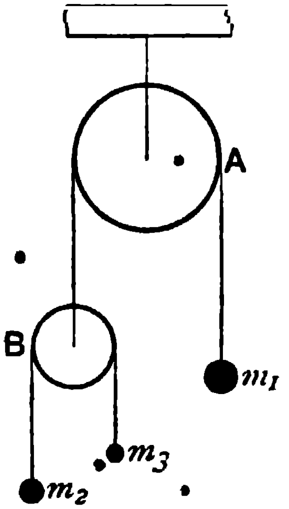
also, $x = e(\sqrt{u^2 \sin^2 \alpha - 2gx})t_2 + \frac{1}{2} g t_2^2$ which gives t_2;

Thus, $T = t_1 + t_2$ . . . . (2.)

Hence by (1) and (2) the condition must be that—

$$t_1 + t_2 = \frac{x(1+e+e^2)}{e^2 u \cos \alpha}.$$

Example xiv. A mass m_1 is made fast to a light string which passes over a fixed pulley A, and the other end is attached to a light pulley B. Again, over B a second light string passes, having at its ends two masses m_2 and m_3 $(m_2 > m_3)$. Find the accelerations of the several masses, and also the tensions of the strings.



Let a_1 = acceleration of m_1 in space.
$a_2 =$,, ,, m_2 and m_3 relatively to B.
Then since m_2 is descending and m_3 is ascending relatively to B;
$\therefore$ $a_1 - a_2$ = acceleration of m_2 in space;
also, $a_1 + a_2 =$,, ,, m_3 ,,
Let T_1 = Tension in string passing over A;
and $T_2 =$,, ,, ,, B.

Consider m_1 $\therefore$ $m_1 a = m_1 g - T_1 \ldots\ldots$ (1.)
,, m_2 $\therefore$ $m_2(a_1 - a_2) = T_2 - m_2 g \ldots\ldots$ (2.)
,, m_3 $\therefore$ $m_3(a_1 + a_2) = T_2 - m_3 g \ldots\ldots$ (3.)
and $T_1 = 2T_2 \ldots\ldots$ (4.)*

These four equations are sufficient to determine

$$a_1,\ a_2,\ T_1,\ T_2.$$

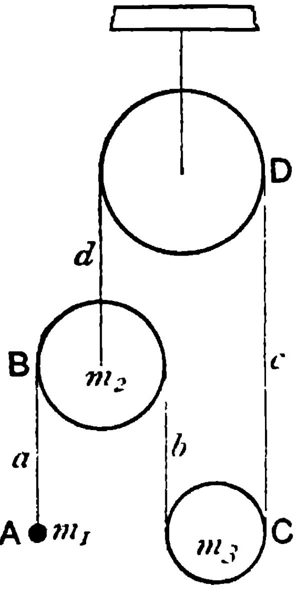
Example xv. A mass m_1 is made fast to one end of an inextensible string which passes over a moving pulley B (mass m_2), then under a second moving pulley C (mass m_3), next over a fixed pulley D, and finally has its other end attached to B. To find the *tension* of the string during the motion, and the *accelerations* of the several masses.

Let us suppose, for simplicity, that all the masses are descending,

m_1 with acceleration $= a_1$
m_2 ,, ,, $= a_2$
m_3 ,, ,, $= a_3$

And let the Tension of the string $= T$.

Consider m_1, $m_1 a_1 = m_1 g - T \ldots\ldots$ (1.)
,, m_2, $m_2 a_2 = m_2 g + 2T - T \ldots$ (2.)
,, m_3, $m_3 a_3 = m_3 g - 2T \ldots\ldots$ (3.)



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* The last equation may not be evident at first sight. Consider what will happen if B have a mass m. Then knowing that its acceleration must be a_1 we have—

$$m a_1 = T_1 - (mg + 2T_2).$$

But since by hypothesis $m = 0$ $\therefore$ $T_1 = 2T_2$.

Now if the motion last for t units of time, then measuring from a fixed horizontal line,

The string d has increased by $\frac{1}{2}a_2t^2$,
„ c „ „ $\frac{1}{2}a_3t^2$,
„ b „ „ $\frac{1}{2}a_3t^2-\frac{1}{2}a_2t^2$,
„ a „ „ $\frac{1}{2}a_1t^2-\frac{1}{2}a_2t^2$;

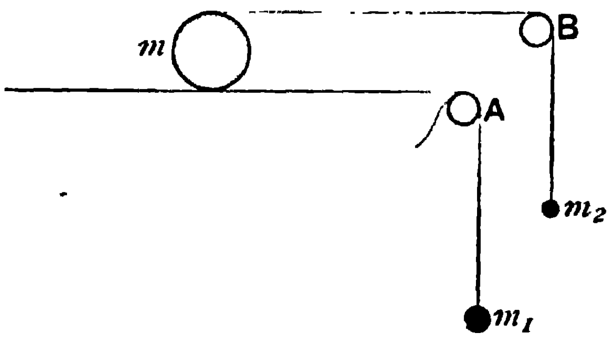
∴ the whole string has increased by $(2a_3+a_1-a_2)\frac{1}{2}t^2$.

But as the string is inextensible, this increase $=0$,

$$\therefore\ 2a_3+a_1-a_2=0 \ldots\ldots\ldots\ldots (4.)$$

The 4 equations 1, 2, 3, 4 are sufficient to determine the unknown quantities a_1, a_2, a_3 and T.

Example xvi. An inextensible string having masses m_1 and m_2 made fast to its ends passes over smooth fixed pegs A and B, and also round a heavy pulley (mass m) on a smooth table. To find the *accelerations* of the several masses and the *tension* of the string, all the bodies moving in a vertical plane.



Let m_1 descend relatively to m_2;

Let the acceleration of $m_1=a_1$.
And „ „ $m=a$.
And let the tension of the string $=T$.

As m advances on the table, the total length of the string which slips over A and B in the time t must $=2\cdot\frac{1}{2}at^2$;

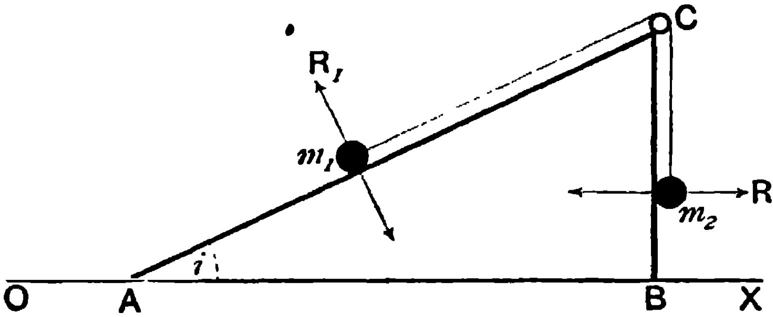
But the length which slips over $A=\frac{1}{2}a_1t^2$;
∴ „ „ „ $B=2\cdot\frac{1}{2}at^2-\frac{1}{2}a_1t^2$,
or $(a-\frac{1}{2}a_1)t^2$;

∴ the acceleration of m_2 downwards $=2(a-\frac{1}{2}a_1)$,
$=(2a-a_1)$.

Consider m_1, $m_1a_1=m_1g-T \ldots\ldots\ldots\ldots (1.)$
„ m_2, $m_2(2a-a_1)=m_2g-T \ldots\ldots\ldots\ldots (2.)$
„ m, $ma=2T \ldots\ldots\ldots\ldots (3.)$

The 3 equations 1, 2, 3, are sufficient to determine a, a_1, and T.

Example xvii. Two masses m_1 and m_2 are connected by a string over the vertex of a wedge (mass m) placed on a smooth table, m_1 being on the slope of the wedge and descending relatively to m_2. To find (1) the acceleration of m_1 and m_2; (2) the acceleration of the wedge; (3) the tension of the string; (4) the pressures of m_1 and m_2 on the wedge; (5) the pressure on the table.



Let a_1 = Acceleration of m_1 and m_2;
a = ,, ,, wedge;
T = Tension of string;
R_1 = Pressure of m_1 on wedge;
R_2 = ,, ,, m_2 ,,
R = ,, on table.

Consider the motion of m_1—

Resolving all the forces acting on m_1 along the plane:

$$m_1(a_1 - a\cos i) = m_1 g \sin i - T \quad \ldots\ldots\ldots \quad (1.)^*$$

Resolving at right angles to the plane:

$$m_1 a \sin i = m_1 g \cos i - R_1 \quad \ldots\ldots\ldots\ldots \quad (2.)$$

Consider the motion of m_2—

Resolving all the forces acting on m_2 horizontally:

$$m_2 a = R_2 \quad (3.)$$

Resolving vertically:

$$m_2 a_1 = T - m_2 g \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.)$$

Consider the motion of the wedge—

Resolving all the forces acting on it horizontally:

$$ma = R_2 + T\cos i - R_1 \sin i \quad \ldots\ldots\ldots\ldots \quad (5.)$$

Resolving all the forces vertically:

$$m \cdot 0 = mg + T + T\sin i + R_1 \cos i - R \quad \ldots\ldots \quad (6.)$$

These 6 equations are sufficient to determine a, a_1, T, R_1, R_2, and R.

---

* To make this equation clear, consider the motion of m_1 *relative to the point of the wedge in contact with it*: the accel. of m_1 down the plane is a_1 and the accel. of the point in question *up* the plane is the component of a in that direction; ∴ accel. of m_1 in space parallel to the plane must be $(a_1 - a\cos i)$ and the force acting in that direction is $mg \sin i - T$.

Example xviii. If the Unit of Work be 500 foot-pounds, the Unit of Force be the weight of 10 lbs. ($g=32\cdot2$ f.-s.-s.), the Unit of Mass be 161 lbs.; required the Units of Space and Time employed in that system.

Let x feet = Unit of Space; and t seconds = Unit of Time.

By definition, Unit of Work = Unit of Force × Unit of Space;

$\therefore$ 500 ft.-lbs = 10 lbs. wt. × x feet,

But, 500 ft.-lbs. = 10 lbs. wt. × 50 feet;

$\therefore x=50$ feet.

Again, in the equation, $F=ma$, if $F=1$, and $m=1$, then a must also = 1.

$$\text{Now, } 10g=161a, \therefore a=\frac{10\times32\cdot2}{161}=2 \text{ f.-s.-s.};$$

$\therefore$ Unit of Acceleration = 2 f.-s.-s.

By definition—

Unit of Acceleration = Unit of Velocity added in Unit of Time.

= a vel. of x feet per t secs. added per t secs.

= a vel. of $\frac{50}{t^2}$ f.-s.-s.;

$$\therefore 2=\frac{50}{t^2}; \therefore t^2=25;$$

$\therefore t=5$ seconds.

Hence, Unit of Space = 50 feet.

Unit of Time = 5 seconds.

APPENDIX II.

DUCHAYLA'S PROOF OF THE PARALLELOGRAM OF FORCES.

We assume the following:—

1°. If a force act on a *rigid* body it will produce the same effect if supposed to act at any point of the body situated in the line of action of the force. (Art. 172.)

2°. The resultant of two forces acting at a point lies between the forces and in the plane of the forces; for it is evident that if any reason can be assigned why it should act on one side of the plane, an identical reason may be assigned why it should act on the opposite side also.

3°. The line of action of the resultant of two *equal* forces acting at a point bisects the angle between them; for it is evident that if any reason can be assigned why it should act nearer to one of the forces, an identical reason may be assigned why it should act nearer the other also.

Statement of the Theorem.

If two forces acting at a point be represented in both direction and magnitude by the adjacent sides of any parallelogram, their resultant is represented in both direction and magnitude by the diagonal of the parallelogram which passes through their intersection.

From 2° and 3° it follows that if two *equal* forces acting at a point be fully represented by the adjacent sides of a rhombus or square, the *direction* of their resultant is along the diagonal which passes through their intersection.

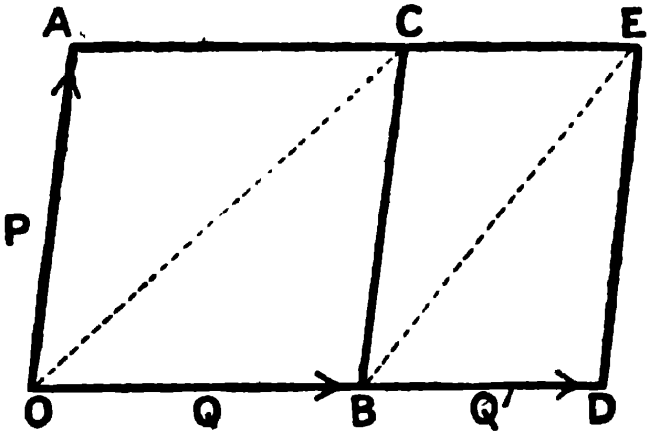
The Proof of the Theorem in the case of unequal forces is divided into three parts:—

(i.) The diagonal represents the *direction* of the resultant, in the case of two *commensurable* forces.

(ii.) The diagonal represents the *direction* of the resultant, in the case of two *incommensurable* forces.

(iii.) The diagonal represents the *magnitude* of the resultant in both cases.

To prove (i.)



If the Theorem (so far as *direction* is concerned) holds for the system P and Q acting at a given angle, and also holds for the system P and Q' acting at the same angle, then we can show that it will also hold for the system P and $Q+Q'$ acting at that angle.

Let OA fully represent P, and OD fully represent $Q+Q'$.

Also let OB fully represent Q acting at O.

Then BD will fully represent Q' acting at B. (1°)

Complete the parallelograms BA and DC.

Join OC and BE.

By hypothesis, the resultant of P and Q acting at O acts along OC.

Replace P and Q by their resultant, and transfer its point of application from O to C, (1°).

At C resolve this resultant into its original components, viz. :—

P, which will act along BC, and may therefore be supposed to act at B, (1°), and

Q, which will act along CE, and may therefore be supposed to act at E, (1°).

We have, now, a force Q acting at E in its original direction; and two forces acting at B, viz. P fully represented by BC and Q' fully represented by BD.

By hypothesis, the resultant of P and Q' acting at B acts along BE.

Replace P and Q' by their resultant, and transfer its point of application to E, (1°).

At E resolve this resultant into its original components, P and Q'.

We have thus transferred the forces acting at O to the point E (supposed to be rigidly connected with the body) without changing their direction or effect.

$\therefore$ *E is a point in the line of action of their resultant.*

$\therefore$ OE is the direction of the resultant of the system P and $Q+Q'$ acting at O.

Now let each force in the system be equal to F.
∴ If the theorem (so far as *direction* is considered) is true
for the two systems, F, F and F, F,
it will be true for the system F and $2F$.
But we have shown that it *is* true for the system F, F;
∴ it is true for the system F and $2F$.
Again, since it is true for F and $2F$, and for F and F,
∴ it is true for the system F and $3F$;
and so on;
∴ it is true for the system F and mF,
where m is a positive integer.
Again, since it is true for mF and F, and for mF and F,
∴ it is true for the system mF and $2F$.
Since it is true for mF and $2F$, and for mF and F,
∴ it is true for the system mF and $3F$;
and so on;
∴ it is true for the system mF and nF,
where both m and n are positive *integers*;
∴ in the case of two *commensurable* forces,
the diagonal represents the *direction* of their resultant.

To prove (ii.)

If the two forces are *incommensurable*, it is always possible to find two *commensurable* forces which will represent the given forces to any degree of closeness;* and since we have shown that the Theorem is true for any two commensurable forces (i.), therefore it must be also true for any two incommensurable forces.

---

* Thus, let P and Q be two incommensurable forces, and let F be any force which *we may take as small as we like.*

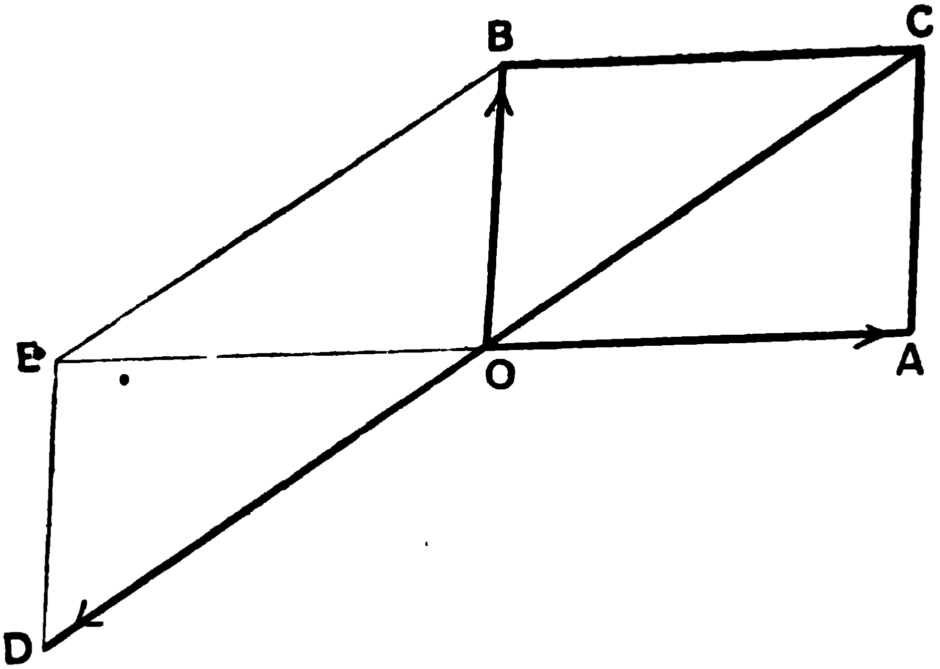
Then, let $\frac{P}{F}=m+k$, and $\frac{Q}{F}=n+k'$,

where m and n are positive *integers*, and k and k' are positive *fractions*;

∴ $P-mF=kF$, and $Q-nF=k'F$.

Now let F become smaller and smaller, then the second sides can be made as small as we like; ∴ P and Q can be made to differ from mF and nF by as small a quantity as we like.

To prove (iii.)



Let OA and OB fully represent *any* two forces acting at O.
Complete the parallelogram AB.
Then it has been shown that OC represents their resultant in *direction*.
Produce CO, and measure off on the produced line a distance OD which shall represent their resultant in *magnitude*.
Then, by the definition of a resultant, the three forces fully represented by OA, OB, OD, are in equilibrium.
Complete the parallelogram DB, and join OE.
Then OE represents the *direction* of the resultant of the two forces OB and OD;
$\therefore$ OA and OE are in the same straight line;
$\therefore$ EC is a parallelogram.
$\therefore$ $OC=EB$; and $OD=EB$; $\therefore$ $OC=OD$.

By construction OD represents the magnitude of the resultant of the two forces OA and OB.

$\therefore$ OC represents the resultant of OA and OB in *magnitude.*

And this completes the Proof.

THE PATH OF A PROJECTILE IS A PARABOLA.

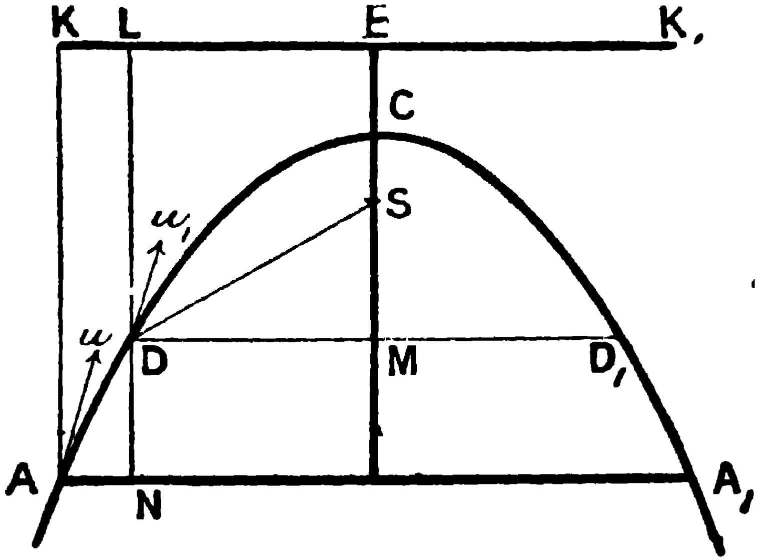
(*Note.*—The student is supposed to have read Arts. 277-292.)

First Method, requiring only the definition of a Parabola.

DEF.—*A Parabola is a plane curve, every point in which is equidistant from a certain fixed point and from a certain fixed straight line.*

The fixed point is called the **Focus.**

The fixed straight line is called the **Directrix.**



Let a particle be projected from any point A with a velocity u at an angle of elevation α

From A draw a vertical straight line AK, and make

$$AK=\frac{u^2}{2g}.$$

Through K draw a horizontal straight line KK_1 of indefinite length. After any time t let the particle be at D, and let its velocity at this point be u_1, making an angle θ with the horizon.

The Hor. Vel. of $u=u\cos\alpha$, and the Vert. Vel. $=u\sin\alpha$.

The Hor. Vel. of $u_1=u_1\cos\theta=u\cos\alpha$. (Art. 281.)

The Vert. Vel. of $u_1=u_1\sin\theta=u\sin\alpha-gt$. (Art. 281.)

$$\begin{aligned}\therefore\ u_1^2&=(u\cos\alpha)^2+(u\sin\alpha-gt)^2\\&=u^2\cos^2\alpha+u^2\sin^2\alpha-2\,ug\sin\alpha.t+g^2t^2\\&=u^2-2ug\sin\alpha\cdot t+g^2t^2\\&=u^2-2g\,(u\sin\alpha\cdot t-\tfrac{1}{2}\,gt^2).\end{aligned}$$

But $DN=u\sin\alpha\cdot t-\frac{1}{2}gt^2$;

$$\therefore\ u_1^2=u^2-2g\cdot DN.$$

$$\text{And } u^2=2g\cdot AK; \qquad \text{(by const.)}$$

$$\begin{aligned}\therefore\ u_1^2&=2g\cdot AK-2g\cdot DN\\&=2g\,(AK-DN)\\\therefore\ u_1^2&=2g\cdot DL.\end{aligned}$$

Therefore, the *total* velocity of the projectile at any given point of its path is equal to that due to its fall from rest from the point in KK_1 vertically over the given point.*

$$\text{Hence } DL=\frac{u_1^2}{2g}.$$

Let C be the highest point reached by the projectile.

Draw ECM a vertical straight line through C, meeting the horizontal line through D at M.

Make $CS=CE$.

Then the total velocity at C is $u\cos\alpha$.

---

* This is usually expressed by saying that the velocity of a projectile at any given point is equal to that acquired in falling vertically from rest in the directrix to the given point.

Hence it follows that

$$CE=\frac{(u\cos\alpha)^2}{2g},\text{ or}=\frac{u_1{}^2\cos^2\theta}{2g}.$$

Now $MS=ME-SE=DL-2CE.$

$$\therefore\ MS=\frac{u_1{}^2}{2g}-\frac{2u_1{}^2\cos^2\theta}{2g}.$$

Let T_1 = Time of Flight from D to D_1;

$$\therefore\ T_1=\frac{2u_1\sin\theta}{g}.\quad\text{(Art. 285.)}$$

$$\therefore\ DM=u_1\cos\theta\times\tfrac{1}{2}T_1=u_1\cos\theta\cdot\frac{u_1\sin\theta}{g}.$$

$$=\frac{u_1{}^2\cos\theta\sin\theta}{g}.$$

Now $DS^2=DM^2+MS^2$

$$=\left(\frac{u_1{}^2\cos\theta\sin\theta}{g}\right)^2+\left(\frac{u_1{}^2}{2g}-\frac{2u_1{}^2\cos^2\theta}{2g}\right)^2.$$

$$=\frac{u_1{}^4}{4g^2}\left(4\cos^2\theta\sin^2\theta+1-4\cos^2\theta+4\cos^4\theta\right)$$

$$=\frac{u_1{}^4}{4g^2}\left\{4\cos^2\theta(\sin^2\theta+\cos^2\theta)+1-4\cos^2\theta\right\}$$

$$=\frac{u_1{}^4}{4g^2}\left(4\cos^2\theta+1-4\cos^2\theta\right)$$

$$=\frac{u_1{}^4}{4g^2}$$

$$\therefore\ DS=\frac{u_1{}^2}{2g}.$$

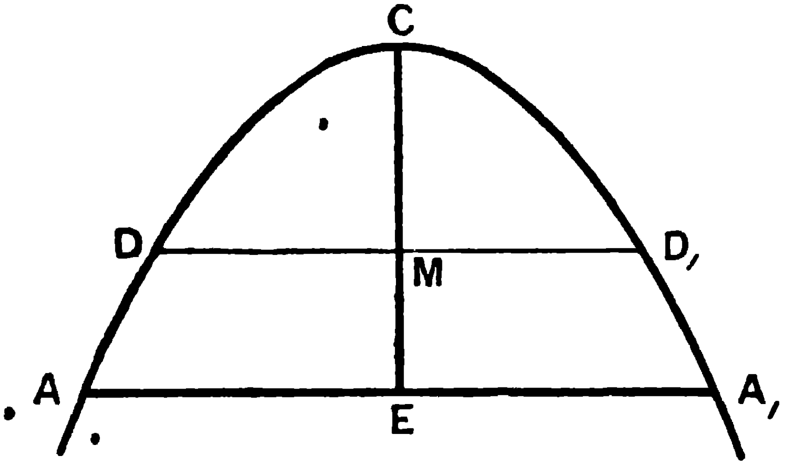
But $DL=\dfrac{u_1{}^2}{2g}$;

$$\therefore\ DS=DL.$$

Now S is a fixed point, and KK_1 is a fixed straight line, and since D may be any point in the projectile's path, ∴ the locus of the point D is a parabola, having S as its focus, and KK_1 as its directrix.

∴ *The Path of a Projectile is always a Parabola.*

Second Method, requiring a knowledge of Conic Sections.



Let a particle be projected from any point A with a velocity u at an angle of elevation α.

After any given time t let its position be at D.

The Hor. Vel. at $D = u \cos \alpha$.

The Vert. Vel. at $D = u \sin \alpha - gt$. (Art. 281.)

Let T_1 = Time of Flight from D to D_1.

$$\text{Then } T_1 = \frac{2(u \sin \alpha - gt)}{g}; \text{ (Art. 285.)}$$

$$\therefore DD_1 = u \cos \alpha \cdot T_1 = u \cos \alpha \cdot \frac{2(u \sin \alpha - gt)}{g}.$$

Let C be the highest point reached by the projectile.

$$\text{The Time of Flight from } D \text{ to } C = \tfrac{1}{2} T_1 = \frac{u \sin \alpha - gt}{g}.$$

$$\therefore DM = u \cos \alpha . \tfrac{1}{2} T_1 = u \cos \alpha \frac{(u \sin \alpha - gt)}{g}$$

$$\therefore DM = \tfrac{1}{2} DD_1.$$

To find CM we use the Formula of Reference,

$$s=ut+\tfrac{1}{2}at^2;$$

$$\therefore\ CM=(u\sin\alpha-gt).\frac{u\sin\alpha-gt}{g}-\tfrac{1}{2}g\left(\frac{u\sin\alpha-gt}{g}\right)^2$$

$$=\frac{(u\sin\alpha-gt)^2}{g}\left(1-\tfrac{1}{2}\right)$$

$$=\frac{(u\sin\alpha-gt)^2}{2g}.$$

Now $DM^2=\dfrac{u^2\cos^2\alpha(u\sin\alpha-gt)^2}{g^2}$

$$=\frac{u^2\cos^2\alpha.\,2gCM}{g^2},$$

$$\therefore\ DM^2=\frac{2u^2}{g}\cos^2\alpha.\,CM.$$

But when the origin is at the vertex, the equation to the Parabola is, $y^2=4ax$; where $4a$ is the latus rectum;

$\therefore$ the locus of D is a Parabola,
whose Vertex is at C, whose Axis is CE,
and whose Latus Rectum is $\dfrac{2u^2\cos^2\alpha}{g}$.

$\therefore$ *The Path of a Projectile must always be a Parabola*

TWO TRIGONOMETRICAL FORMULÆ.

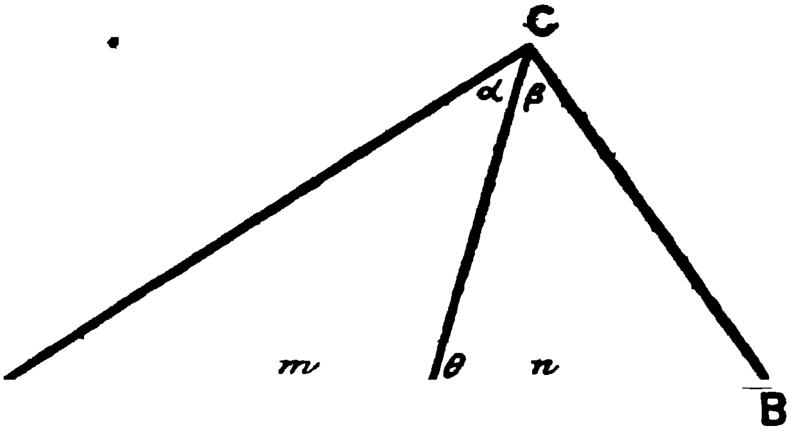
The following Theorems are very often of great use in solving statical problems—

If ABC be any plane triangle, and from C a straight line CD be drawn to any point in the base; then if $AD=m$; $DB=n$; $ACD=\alpha$; $BCD=\beta$, and $BDC=\theta$, we shall always have

$$(m+n)\cot\theta=m\cot\alpha-n\cot\beta\quad .\ .\ .\quad (1)$$

and, $(m+n)\cot\theta=n\cot A-m\cot B\quad .\ .\ .\quad (2)$

To prove (1).



In the triangle ACD,

$$\frac{AD}{AC}=\frac{\sin\alpha}{\sin(\pi-\theta)}=\frac{\sin\alpha}{\sin\theta}.$$

In the triangle BCD,

$$\frac{DB}{BC}=\frac{\sin\beta}{\sin\theta};$$

$$\therefore\ \frac{AD}{AC}\times\frac{BC}{DB}=\frac{\sin\alpha}{\sin\theta}\times\frac{\sin\theta}{\sin\beta};$$

$$\therefore\ \frac{m}{n}\times\frac{a}{b}=\frac{\sin\alpha}{\sin\beta},$$

$$\therefore\ \frac{m}{n}\times\frac{\sin A}{\sin B}=\frac{\sin\alpha}{\sin\beta}.$$

But $A=\theta-\alpha$; and $B=\pi-(\theta+\alpha)$;

$$\therefore\ \frac{m\sin(\theta-\alpha)}{n\sin(\theta+\beta)}=\frac{\sin\alpha}{\sin\beta};$$

$$\therefore\ m\sin\beta\sin(\theta-\alpha)=n\sin\alpha\sin(\theta+\beta);$$

$$\therefore\ m\sin\beta(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$$
$$=n\sin\alpha(\sin\theta\cos\beta+\cos\theta\sin\beta).$$

Divide every term by $\sin\alpha\sin\beta\sin\theta$;

$$\therefore\ m(\cot\alpha-\cot\theta)=n(\cot\beta+\cot\theta)$$

$$\therefore\ (m+n)\cot\theta=m\cot\alpha-n\cot\beta.$$

Q. E. D.

To prove (2).

$$\text{As before, } \frac{m \sin A}{n \sin B} = \frac{\sin \alpha}{\sin \beta}.$$

$$\text{But } \alpha = \theta - A\,; \text{ and } \beta = \pi - (\theta + B)\,;$$

$$\therefore \ \frac{m \sin A}{n \sin B} = \frac{\sin(\theta - A)}{\sin(\theta + B)}\,;$$

$$\therefore \ m \sin A \sin(\theta + B) = n \sin B \sin(\theta - A)\,;$$

$$\therefore \ m \sin A (\sin\theta \cos B + \cos\theta \sin B)$$
$$= n \sin B (\sin\theta \cos A - \cos\theta \sin A).$$

Divide every term by $\sin A \sin B \sin\theta$;

$$\therefore \ m(\cot B + \cot\theta) = n(\cot A - \cot\theta)$$
$$\therefore \ (m+n)\cot\theta = n \cot A - m \cot B.$$

Q. E. D.

The student may, as an exercise, employ the first of these results in working the following :—

Page 189, No. 7 ; page 193, Nos. 17, 24 ; page 194, No. 34 ; page 337, No. 106.

EXAMPLES FROM RECENT EXAMINATION PAPERS.

THE student ought to know that in nearly all Examination Papers in this subject there is a note at the beginning to the effect that *great importance is attached to accuracy.*

I. COLLEGE OF PRECEPTORS.

1. The wind blows from a point intermediate between N. and E. The northerly component of its velocity is 10 miles an hour, and the easterly component is 36 miles an hour. Find the velocity of the wind.

2. Explain what is meant by the *unit of force*, taking as units (1) the centimetre, second, and gramme, (2) the foot, second, and pound.

3. A mass of 20 lbs. is projected down a rough inclined plane (ratio of *height* to length $= 3 : 5$) with a velocity of 30 f.-s. If $\mu = \cdot 2$, find the velocity of the body when it has traversed a distance 200 feet along the plane.

4. Forces 3, 4, 5, 6 units act on a particle at the centre of a square in directions towards the angular points respectively. Find the resultant force.

5. A uniform beam, weight W and length l, rests with one end against a smooth vertical wall, and with the other end on a smooth horizontal floor, and connected with the base of the wall by a string, length s. Find the tension in the string, and the reactions at the ends of the beam.

6. A train starts from rest, and acquires a velocity of 60 miles per hour in 10 minutes. What is the measure of its acceleration, taking as units (1) miles and minutes, (2) feet and seconds?

7. What is the least force necessary to cause a mass of 15 lbs. to move through 20 feet from rest in $2\frac{1}{2}$ seconds?

8. A solid cube rests on a rough horizontal plane; through what angle may the plane be inclined before the cube overturns?

II. INTERMEDIATE EXAMINATION (Ireland).

9. A man, 10 stones mass, goes up a ladder 16 feet high; what work does he do in raising himself?

10. Show, by the 'Principle of Work,' how the mechanical advantage of the single movable pulley can be ascertained?

11. How is it proved experimentally that all bodies fall with equal rapidity in a vacuum?

12. Explain how the relations between the time, velocity, and space described by a moving particle can be represented by a geometrical construction.

13. If brakes were applied to all the wheels of a train moving at 30 miles an hour on a level, within what space would it be pulled up, the co-efficient of friction being $\frac{1}{8}$?

14. A heavy uniform plank ABC, weighing 60 lbs., and 14 feet long, rests on two props A, B, 12 feet apart. Could a man weighing 140 lbs. walk on it to the end C without the plank toppling over?

15. A string is laid over two pulleys A, B, in the same horizontal line, 10 feet apart; a mass of 112 lbs. is hung at each end; if a mass of 1 lb. be placed on the string half-way between A and B, find how far it will descend below AB when in equilibrium.

16. Two equal masses m and m are connected by a string over a small smooth pulley. A given mass m_1 is now placed on one of them; determine the pressure it exerts on the body with which it is in contact during the motion.

III. OXFORD LOCAL.

17. Define *force*, and give examples of forces of different types.

18. What is the *moment* of force about a point? Show that a moment can be represented geometrically by an area.

19. What is the *centre of gravity* of a body? How would you determine experimentally the C. G. of a flat disc? Clearly explain the reason for each operation.

20. Define *energy*; explain what is meant by the *conservation* of energy, and give an example of this conservation.

21. A mass of 10 lbs. is suspended by a string; and when pulled by a second string in a direction making an angle of 30° below the horizon, the first string makes an angle of 30° with the vertical. Find the tensions in the two strings.

22. Show how to find the C. G. of the remainder of a body after a portion has been cut away, the masses and centres of gravity of the whole and of the portion cut away being known.

23. A uniform beam 20 ft. long, weighing 100 lbs., carries a load of 50 lbs. at a point 6 ft. from one end, and rests on two supports at points distant 8 ft. and 15 ft. from this end. Find the pressures on the supports.

24. Masses of 5 lbs. and 6 lbs. are attached to the ends of a string which passes over a smooth pulley. Find the acceleration with which they will move, and the tension in the string in lbs. weight.

25. Give a definition of *force* in terms which are suggested by the equation $F = md$, and from this equation deduce a convenient unit of force.

How is *weight* expressed in terms of this unit?

26. Prove that a 'couple' cannot be balanced by a single force of translation.

27. A body acquires the speed of 150 miles an hour in 44 seconds when moving with uniform acceleration: how soon will it have reached a place 480 yards from its starting-point?

28. How does the potential energy alter when a body falls through the air to the mouth of a mine and then proceeds onward down the mine?

29. A sphere, of mass 6 lbs., moving with a velocity of 8 f.-s., overtakes another sphere, of mass 4 lbs., moving in the same straight line and in the same direction with a velocity of 4 f.-s.; the coefficient of restitution being $\frac{1}{2}$, find the velocities of the spheres after impact.

30. A ball is projected from the foot of a plane, inclined at an angle of 30° to the horizon, in a direction making an angle of 30° with the plane. Show that the velocity with which it must be projected in order that it may strike the plane at a distance of 20 feet from the foot is $8\sqrt{15}$ feet per second.

31. A ladder is placed on a rough horizontal pavement and leaned against a smooth vertical wall. It is found that the ladder is about to slip when it makes an angle of 45° with the horizontal. Find the coefficient of friction between the ladder and the pavement.

32. A $\frac{1}{2}$-oz. bullet is fired with a velocity of 1400 f.-s. from a gun weighing 7 lbs. Find the velocity in feet per second with which the gun begins to recoil, and the mean force in lbs. weight that must be exerted to bring it to rest in 4 inches.

33. Two forces of 20 lbs. weight and 11 lbs. weight respectively, acting at a point, have a resultant equal to 13 lbs. weight; if the smaller force is increased by 10 lbs. weight, determine the magnitude of the new resultant.

34. An ordinary door 9 feet high and 5 feet wide, and weighing 150 lbs., is supported by two hinges, one 18 inches from the top and the other the same distance from the bottom; draw figures to illustrate the forces acting on the door, and determine their magnitude (1) when the reaction at the upper hinge has no vertical component, (2) when the vertical components of the reactions at the two hinges are equal.

35. A hemispherical bowl 18 inches in diameter and weighing 15 lbs. is suspended from a point by three chains, each of which is 3 ft. 5 in. long, fastened to its rim at three points equidistant from one another. Determine the tension of each chain.

36. Show that the position of the C. G. of a body relatively to the body is independent of the position of the body.

37. On three of the faces of a regular tetrahedron of height h regular tetrahedra are described. Find the distance of the C. G. of the solid thus formed from the fourth face of the original tetrahedron.

38. A heavy body, resting on a rough plane inclined at an angle a to the horizontal, is connected with mass hanging freely by a string passing over the vertex; if m_1 and m_2 are the greatest and least values of this mass consistent with the equilibrium of the system, prove that the coefficient of friction between the plane and the body is

$$\frac{(m_1-m_2)\tan a}{m_1+m_2}.$$

39. The apparent velocities of two ships A and B as seen from a third ship C are represented by two straight lines PQ and PR; how will the apparent velocities of B and C as seen from A be represented?

40. A particle is projected vertically downwards with a velocity u, and t seconds afterwards another body is projected from the same point in the same direction with a velocity u_1 which is greater than u; prove that the second body will never overtake the first unless $t < \frac{u_1-u}{g}$.

41. A heavy mass m is being raised vertically by means of a fine string; determine the tension of the string, when the mass is rising (1) with a uniform velocity v; (2) with a uniform acceleration a.

42. In $6\frac{1}{4}$ seconds after projection the velocity of a body is $\frac{3}{4}$ths of its initial velocity, and it is moving at right angles to the direction in which it was projected. Determine the magnitude and direction of its initial velocity. ($g=32$.)

IV. CAMBRIDGE LOCAL.

43. Explain the principle of transmissibility of force, and show how the principle is used in proving the 'Parallelogram of Forces.'

Two weightless strings are tied to the ends of a uniform bar; each passes over a pulley, and has a weight attached to it equal to the weight of the bar. Show that when in equilibrium the bar is horizontal, and find the inclination of the strings to the vertical.

44. Show that if three forces in one plane be in equilibrium, they all pass through one point, or are all parallel.

A rod rests with one end against a smooth inclined plane, and the other end is held by a string. Give a geometrical construction for finding the direction of the string. What takes place if the plane be horizontal?

45. Find the C. G. of a thin uniform triangular lamina.

Find the C. G. of a quadrilateral lamina, two of whose sides are parallel and one of them double of the other.

46. Show how to graduate the Common Steelyard.

Explain carefully why a man stands on the bottom rung of a ladder, and holds on to another rung as low down as he can, when another man is lifting the ladder.

47. Define Velocity, and prove the 'Parallelogram of Velocities.'

A boat is set with her head due N.E. Under the action of the wind alone the boat would move in a N.E. direction with a velocity of $4\sqrt{2}$ miles per hour. The tide is flowing due South at the rate of 4 miles per hour. Show that the boat's actual course is due East.

48. Assuming only the definition of velocity and of uniform acceleration, prove that the velocity acquired by a particle while moving from rest with a uniform acceleration f for a distance s in a straight line is $\sqrt{2fs}$.

49. A cricket-ball is thrown vertically upwards with a velocity of 56 feet per second. Find the velocity when it is half-way up, and the height to which it has risen when half the time to the highest point has elapsed. (The resistance of the air is neglected, and the acceleration of gravity = 32 feet per second each second.)

50. Prove the 'Parallelogram of Forces' for two incommensurable forces.

Show how to find the resultant of two forces represented by the diagonal of a cube and one of the edges meeting the diagonal, both acting from the point where they meet. Prove that its magnitude $= \sqrt{6} \times$ an edge.

51. Find the resultant of two unlike parallel forces.

Show that the resultant acts towards the side of the greater of the two forces, and can never act between them. What happens if the forces become equal?

52. Having given the C. G. of a body, and that of a portion of it, find the C. G. of the remaining portion.

Find the C. G. of the remaining portion of a parallelogram, when a triangle has been cut off from the parallelogram by a single straight cut.

53. Describe the construction of the common balance, stating the requisites of a good balance. Is it possible to secure all those requisites in the same balance?

A balance consists of a uniform rod, of length 18 inches, and weight $= \frac{1}{2}$ lb., the fulcrum being one-eighth of an inch to one side of the C. G. of the rod. If a 1 lb. weight be in the scale attached to the shorter arm, find how much the customer has weighed out to him in the other scale.

54. A man starts at right angles to the bank of a river, at the uniform rate of $1\frac{1}{4}$ miles per hour, to swim across; the current for part of the way is flowing uniformly at the rate of 1 mile per hour, and for the remainder of the way at double that rate. He finds when he reaches the other side that he has drifted down the stream a distance = the breadth of the river. At what point did the speed of the current change?

55. Show that the difference of the squares of velocities at any two points, of a body falling in vacuo, varies as the distance between them.

A body falls from rest in vacuo through a certain height, and acquires a certain velocity. Find how much farther the body will have fallen when it has doubled its velocity.

56. Describe an apparatus for verifying experimentally the 'Parallelogram of Forces.'

Forces of 2, 3, and 4 lbs. act at a point O in directions parallel to the sides AB, AC, BC of an equilateral triangle respectively; find their resultant.

57. A heavy uniform beam of length 7 feet rests horizontally on two supports, one at one end and the other $5\frac{1}{2}$ feet from that end; if the greatest weight that can be hung on the other end of the beam without disturbing the equilibrium be 16 lbs., find the weight of the beam.

58. What is meant by mechanical advantage? Explain the principle of work as applied to a machine.

Find the mechanical advantage gained by the use of a single moveable pulley when the strings are parallel.

59. A stone is thrown vertically upwards with a velocity of 36 feet per second. To what height will it rise, and after what intervals of time will it have a velocity of 12 feet per second?

60. 'The weight of a body is proportional to its mass.' Examine the evidence on which this statement rests. With what limitations is it true?

Describe and explain some method of determining the value of the acceleration due to gravity at the earth's surface.

61. A projectile weighing half a ton is fired with a velocity of half a mile a second from a 100-ton gun. Find the velocity of recoil of the gun, and compare its kinetic energy with that of the projectile.

62. Show that a force may be resolved into two components in any number of different ways, and explain what is meant by the resolved part of a force in any given direction.

A straight line *COB* has a line *OA* at right angles to it, and forces, each of 7 lbs. weight, act, one along *OA*, another along *OB*, and a third along the bisector of the angle *COA*. Find the magnitude of the resultant.

63. A uniform bar 3 feet long and weighing 5 lbs. rests on a horizontal table with one end projecting 4 inches over the edge; find the greatest weight that can be hung on the end without making the bar topple over.

64. State the laws of friction, and explain what is meant by 'limiting friction.'

A body of weight 10 lbs. rests on a rough plane inclined at an angle of 30° to the horizontal; find the force of friction required to sustain it, and the least value of the coefficient of friction necessary to prevent slipping.

65. A stone is dropped from a height of 8 feet above the ground from the window of a railway carriage travelling at the rate of 15 miles per hour; find its velocity on striking the ground.

66. Define the term Force, and explain how forces are measured by the accelerations they produce.

Two masses of 3 and 4 pounds respectively are connected by a string over a pulley. Find the acceleration and tension of the string.

67. A 50-ton engine moving at the rate of 10 miles per hour impinges on a truck at rest weighing 10 tons, and the two move on together. Find their velocity and calculate the loss of K.E.

68. State and prove the proposition known as the 'Triangle of Forces.'

If three forces are represented in magnitude, direction, *and lines of action* by the sides of a triangle taken in order, what would be the nature of their resultant?

69. Find the resultant of two parallel forces acting in the same direction.

Weights of 1 and 5 lbs. are fixed at the two extremities of a uniform heavy bar 3 feet long. The centre of gravity of the whole is 1 foot from one end. Find the weight of the bar.

70. Find the horizontal force required to sustain a body of weight W on a smooth plane inclined at an angle α.

A ladder rests on a rough horizontal plane and against a smooth vertical wall. Give a geometrical construction for the total reaction of the plane. If a man ascends the ladder, at what stage is it most likely to slip.

71. If a body is moving with a velocity which is not uniform, show how the space passed over in any time may be represented by the area of a diagram. Deduce the formula for the space described in time t from rest, when the velocity is uniformly accelerated.

72. Describe Atwood's machine, and explain how it is used to measure the acceleration of gravity.

Two equal weights of 1 lb. each are connected by a fine string passing over a light pulley. A weight of one ounce is attached to one of them. If $g=32$, find the acceleration and the tension of the string.

73. A ball, with coefficient of elasticity $e=\frac{3}{4}$, is let fall to the ground from a height of 32 feet. If $g=32$, find the loss of K. E. on impact, and the height to which the ball will rebound.

The following are from the Papers set to Senior Students.

74. $ABCD$ is a square: F is the middle point of AB, and E the middle point of BC. Find the direction and magnitude of the resultant of forces acting at B parallel and proportional to AE, ED, DF, and FC.

75. Find the C. G. of a portion of a uniform circular wire which subtends an angle θ at the centre.

From a uniform square lamina a semicircular piece, defined by a circle described on one side as diameter, is cut out. Find the C. G. of the remainder.

76. Supposing the side of a hill a quarter of a mile long to be plane, smooth, and inclined at an angle of 30° to the horizon, find the velocity acquired by a stone slipping from the top to the bottom of the hill.

77. State and explain Newton's Second Law of Motion.

How is 'change of velocity' to be estimated? If a velocity represented in all respects by the side AB of a triangle ABC be changed into a velocity represented in all respects by the side BC, show how to draw a line representing the change of velocity in all respects.

78. A picture hangs by a cord passing over a nail in the usual way. Find the tension of the cord in terms of the angle which either part makes with the horizon and of the weight of the picture, and explain carefully why the tension is decreased by lengthening the cord.

79. The point O is the C. G. of three unequal weights P, Q, R, placed at points A, B, C, respectively. Show that the areas of the triangles OBC, OCA, and OAB are in the ratio of P to Q to R.

80. A uniform rod AB, inclined at an angle of 30° to the horizon, rests just on the point of slipping with the end A in contact with a rough horizontal table, the end B being supported by a string attached to a point C vertically above A. If BC be inclined at an angle of 60° to the horizon, find the angle of friction and the tension of the string.

81. Describe experiments to verify the Second Law of Motion in the case of a falling body.

82. The horizontal velocity of a shot is 1100 feet per second and the range 3000 yards; find the initial vertical velocity.

83. Show that if a body fall freely under gravity there is neither loss nor gain of energy; and explain how to apply the same principle to find the velocity of a body sliding down a smooth curve.

84. Show how to find graphically or otherwise the resultant of a number of forces acting on a rigid body at one point, and apply your method to find the resultant of forces, 1 in an easterly direction, $\sqrt{2}$ in a north-easterly direction, and 1 to the north.

85. Show that if a body be suspended from one point, the centre of gravity and the point of support are in the same vertical line; and apply this to find practically the C. G. of a piece of cardboard of irregular shape.

86. A weight rests on a smooth inclined plane. Determine the direction and magnitude of the least possible force which will keep it in equilibrium. Find also the direction of the force in order that the pressure on the plane may be double of that exerted in the first case.

87. Two equal billiard balls, in which $e=1$, impinge directly. Show that they interchange velocities.

88. A fly-wheel is brought to rest after n revolutions by a constant frictional force applied tangentially to its circumference. If k be the K. E. of the wheel before the friction is applied and r its radius, show that the friction is $k/2\pi nr$.

V. ENGINEER STUDENTS AND DOCKYARD APPRENTICES.

89. $ABCDEF$ is a regular hexagon and O is the centre of the circle which circumscribes it: forces $4P$, $5P$, $8P$, P, $7P$, $6P$ act along OA, OB, OC, OD, OE, OF respectively; find the magnitude and direction of their resultant.

90. A uniform triangular plate ABC is suspended (1) by a string attached to A, and (2) by a string attached to B. If the side AB be inclined to the vertical at the same angle during each suspension, prove that the other two sides of the triangle are equal.

91. Find the magnitude and line of action of a single force which produces the same effect as three forces of 3, 4, and 5 respectively, acting along the sides of an equilateral triangle taken in order.

92. If there be n pulleys (including the fixed one) in the Third System, and if the weight of each of them be p, prove that, instead of taking their weights into account, we may suppose the pulleys to be without weight, and an additional weight of $\left(1-\frac{n}{2^n-1}\right)p$ to be added to the power.

93. The side AB of a triangle ABC is bisected at D, and forces act at the point C in the directions CA, CB, CD, proportional to CA, $2CB$, CD respectively. If the straight line CF, which represents the resultant of these forces in direction and magnitude, meet AB at E; find the ratios $AE : BE$ and $CE : CF$.

94. Three uniform wires of the same thickness are jointed at their ends so as to form a triangle. If two of the wires are equal, and each be three times as long as the third wire, find the ratio in which their C. G. divides the line joining the vertex to the middle point of the base of the isosceles triangle.

95. A uniform straight rod, whose weight is 2 lbs., and length is 12 inches, is suspended from a small hook by two strings, whose lengths are 14 inches and 10 inches respectively, attached to the ends of the rod. Prove that the rod may be kept in a horizontal position by suspending a weight of 4 lbs. from one of its ends.

96. A body, whose weight is W, is just kept from sliding down a rough plane, inclined to the horizon at an angle of 30°, by a force $\frac{1}{4}W$ acting along the plane. If this force be removed, find the magnitude of the force which, acting horizontally, will just keep the body from slipping. Find also the coefficient of friction between the body and the plane.

VI. RANK OF LIEUTENANT, ROYAL NAVY.

(STATICS).

97. A triangular lamina ABC is supported at its three angular points, and a weight equal to that of the lamina is placed upon it; if the pressures at the points A, B, C are proportional to $4a+b+c$, $a+4b+c$, $a+b+4c$, where a, b, c are the lengths of the sides, show that the position of the weight is the centre of the inscribed circle.

98. In a weighing machine constructed on the principle of the common steelyard, the *pounds* are read off by graduations reaching from 0 to 14, and the *stones* by weights hung at the end of the arm; if the weight corresponding to one stone be 7 ozs., the moveable weight $\frac{1}{2}$ lb., and the length of the arm one foot, prove that the distances between the graduations are $\frac{3}{4}$ inch.

99. If four forces, acting in the directions AB, AD, CB, CD are in equilibrium, $ABCD$ being a quadrilateral inscribable in a circle, show that each force is proportional to the side opposite to it.

100. A uniform heavy rod, AB, 16 feet long and 48 lbs. in weight, rests with one end B against a smooth vertical wall, while a point C in the rod, 2 feet from B, rests at the same time on a smooth horizontal rail. Find the inclination of the rod to the horizon, and the pressures at B and C.

101. A triangular lamina of uniform thickness is hung horizontally from three vertical chains which are fastened to the middle points of its sides. Find its weight, if a 12-stone man can stand anywhere upon it without tilting it.

102. If a common steelyard weigh 30 lbs., and the distances of its C. G. and of the point of suspension of the weight from the fulcrum on the same side of it be respectively $\frac{1}{2}$ inch and $1\frac{1}{2}$ inches, of how much will a purchaser of 29 lbs. be defrauded if $\frac{1}{4}$ oz. has been fraudulently filed off the moveable weight, which ought to be 2 lbs.?

103. A piece of uniform heavy wire is formed into a triangle ABC, and the middle points of its sides are joined by pieces of wire of the same thickness. If the framework so formed be hung up from the angle A, show that if θ and ϕ be the angles of the sides AB, AC make with the vertical, then

$$(5a+2b+5c)\,c\sin\theta=(5a+5b+2c)\,b\sin\phi.$$

104. When a mass of 40 lbs. is placed in the scale pan of a Danish steelyard, the fulcrum is 2 inches from the free end; and when a mass of 24 lbs. is placed in the scale pan, the fulcrum is 3 inches from the same end. What mass must be placed in the scale pan so that the fulcrum may be 4 inches from the free end when the steelyard is in equilibrium?

105. In the First System of Pulleys, if there be 3 moveable blocks whose weights (beginning from the lowest) are w_1, w_2, w_3 respectively, what force at the 'fall' will sustain a weight W? And show that your result agrees with the Theory of the Conservation of Energy.

106. A rod ABC, 16 inches long, rests in a horizontal position upon two supports at A and B one foot apart, and it is found that the least upward and downward forces applied at C, which would move the rod, are 4 ozs. and 5 ozs. respectively. Find the weight of the rod and the position of its C. G.

107. A triangular board ABC ($B=90°$) can turn in a vertical plane about a hinge at A, and it is kept in a position of equilibrium with the side BC vertical by a horizontal force applied to C. If the weight of the board be 7 ozs., and $AB=2BC$, find the magnitude and direction of the reaction at the hinge.

108. AB is a diameter of a circle, BP and BQ are chords at right angles to each other. If BP, BQ represent forces, prove that their moments about A are equal.

109. In the Third System of Pulleys there are three moveable blocks of weights w_1, w_2, w_3 beginning with the lowest, and the force P at the 'fall' balances a weight W. When the first and second blocks are interchanged, a force P_1 at the 'fall' balances W. Show that

$$P_1-P=\frac{4}{15}(w_1-w_2).$$

110. Four forces (each $=F$) in the same plane act at a point. The angles between the 1st and 2nd, the 2nd and 3rd, the 3rd and 4th, are each 72°. Find the magnitude and direction of their resultant.

111. A cylindrical vessel, whose height is 8 inches and diameter 6 inches, stands upon a horizontal plane, and a smooth rod, 12 inches long, is placed within it and rests against the edge. If the rod weigh $12\frac{1}{2}$ ozs., show that the pressure between its lowest point and the cylinder is equal to the weight of $\sqrt{109}$ ozs.

112. Four particles, weighing respectively 3, 4, 5, and 8 lbs., are placed at the consecutive angles of a square upon a horizontal plane. If their C. G. is distant $7\sqrt{10}$ inches from the centre of the square, find the length of a side of the square.

113. Three forces acting in the same plane keep a body in equilibrium. If the angles between their directions be 135°, 120°, and 105°, compare their magnitudes.

114. Given a force, how would you obtain graphically its components in any two given directions.

Why, in moving heavy weights from the hold of a ship, is the rope attached to the weight sometimes pushed in a horizontal direction?

115. Show that a prism whose section is a triangle ABC, if laid with the face AB on a horizontal table, will not fall over if the vertical line through C meet the table at a distance from the nearer of the points A, B less than AB.

116. A uniform rod AB, centre C, is capable of turning about A, and a string is fastened to C and to a point D vertically over A, so that ACD is an equilateral triangle; find the tension of the string and the reaction at the hinge in terms of the weight (W) of the rod. If, further, a weight equal to the weight of the rod be suspended from B, what changes will be introduced in the tension and the reaction?

117. A uniform rod, of length $2a$, with one end in contact with a *rough* vertical wall, rests over a *smooth* rail at a distance c from the wall. Show that the rod will rest with its end placed anywhere between two limiting positions, and find them.

118. Four equal particles are placed at equal intervals on a quadrant, two of the particles being at the extremities of the arc. Find the position of the C. G. of the four particles.

119. Two equal weights are attached to a string that is laid over the top of a double inclined plane, the inclinations of the sides to the horizon being 30° and 60° respectively. Show that the bodies will be on the point of moving if the coefficient of friction between each plane and the body on it be $2-\sqrt{3}$.

120. On opposite sides of the straight line AB, a perpendicular AC and an equilateral triangle ABD are drawn. Find the magnitude and the inclination to AB of the resultant of forces $P\sqrt{2}$, $P\sqrt{6}$, $2P\sqrt{2}$ acting at A along AC, AB, AD respectively.

121. A uniform rod, which can turn about a hinge at one end, has the other end joined by a weightless string to a point vertically above the hinge at a distance from it equal to the length of the rod. Show that the tension of the string varies as its length, and is equal to the action at the hinge when the rod is horizontal.

122. $ABCD$ is a rectangular lamina whose sides AB, BC are as 7 : 9, and the diagonals intersect at O. If the portion DOC be cut out, and the remainder suspended from A, show that in the position of equilibrium AB, AD will be equally inclined to the vertical.

123. Each of four sailors places a hand on a spoke of a capstan by which an anchor weighing 24 cwt. is to be lowered. If the radius of the capstan is 15 inches, and the distance between its rim and each hand is 6 feet 3 inches, what is the strain upon the arm of each sailor when the anchor is just about to move?

124. Two children, each of weight W, are swinging on a see-saw, formed by placing a plank, of weight w, across a horizontal cylinder of radius c. When the plank has been turned through an angle θ with the horizon, find the moment of the forces tending to turn it back.

Also prove that the greatest angle through which the plank can swing without slipping is double of the angle of friction between the plank and the cylinder.

125. The lengths of the sides of a quadrilateral are proportional to 1, 3, 5, 6. Forces of 2, 6, 9, 12 respectively act at a point in directions parallel to the sides of the polygon taken in the above order. Determine their resultant.

126. Masses 5, 4, 6, 2, 7, 3 are placed at the corners A, B, C, D, E, F of a regular hexagon. Find the position of their C. G.

127. A uniform wire is bent into the form of a triangle whose sides are 12, 16, and 20 inches. It is suspended from a point O in the side whose length is 16, and when in equilibrium it is found that the line joining the greatest angle with the middle point of the opposite side is vertical. Determine the position of O.

128. A uniform rod rests against a smooth vertical wall with its other extremity in a small hemispherical cup. Prove that the angle which the radius from the centre of the cup to the point of contact of the rod with the cup makes with the vertical $=\tan^{-1}\left(\frac{b}{2h}\right)$, where b is the distance of the cup from the wall, and h is the height above the cup of the point of contact of the rod with the wall.

129. A uniform rod is placed against a vertical wall, at an inclination of $\tan^{-1}(2)$ to the ground; the coefficient of friction between the rod and wall, and between the rod and ground, being in each case $\frac{1}{3}$. Find the position in which a heavy ring, equal in mass to the rod, must be placed on the rod so that the rod may be just about to slip.

130. $ABCD$ is a square whose sides are 2 inches long. On the sides BC, CD, DA, equilateral triangles BEC, CFD, DGA, are constructed, outside the square. Find the distance of the C. G. of the figure $ABECFDGA$ from the C. G. of the square.

131. A uniform rod, AB, which can turn freely in a vertical plane about a hinge at A, has its other extremity supported by a string BC, C being a fixed point in a horizontal line through A. If the weight of the rod be 2 lbs., and $AB=BC=\frac{5}{8}AC$, find the tension of the string and the magnitude of the action at the hinge.

132. Two equal bars, AB, BC, each 1 foot long, and each of weight W, are jointed at B, and suspended by strings OA, OB, OC, each 1 foot long, from a fixed peg O; find the tensions of the strings.

133. Out of a triangular lamina, a triangular piece of given area is cut, having its base on one of the sides of the lamina and its vertex at the C. G. of the lamina. Prove that the C. G. of the remainder of the lamina always lies on the perimeter of a triangle similar to the given triangle.

134. Two equal spheres, of weight W and diameter 4 feet, are connected by a string 50 inches in length, which is laid over two pegs, A and B, in the same horizontal line and 2 feet apart. Find the tension of the string and the pressure between the spheres in the position of equilibrium.

135. A circular disc, of radius a and weight W, is placed within a smooth sphere, of radius b, and a particle of weight w is placed on the disc, the coefficient of friction between it and the disc being μ. Find the greatest distance from the centre of the disc at which the particle can rest.

136. A body rests on a rough plane inclined at an angle i to the horizon, and is kept at rest by a force acting up the plane. It is noticed that any force intermediate in magnitude between P and Q will preserve the equilibrium. Express the weight of the body and the coefficient of friction in terms of P, Q, and i.

(KINETICS.)

137. AOC is the vertical diameter and BOD is the horizontal diameter of a vertical circle. One heavy particle, P, starting from rest at the highest point, A, slides down the chord AB, and another heavy particle, Q, slides down the chord BC, starting from B with a velocity equal to that with which P arrived at B. Find the time P takes to go from A to B, and the time Q takes to go from B to C.

138. A man who weighs p lbs. ascends a feet in a lift. The lift takes t seconds to start and t seconds to stop (the acceleration and retardation being constant), and the rest of the journey is performed with constant velocity v f.-s. Express in pounds weight the pressure of the man on the lift at each instant of the ascent.

139. A small mass is whirled in a vertical circle by a string made fast at one end. The string is a yard long, and it is noticed that the velocity of the body is doubled in passing from its highest to its lowest point. Find the velocity at the instant that the string is horizontal.

140. At a place where $g=32{\cdot}2$ a mass of 200 lbs. falls from a height of 6 feet, and drives a pile half an inch into the ground. Express in poundals the mean pressure on the pile.

141. Two particles describe the perimeter of an equilateral triangle ABC in the same directions and with the same uniform velocity, one starting from A and the other from B at the same instant. Show that one particle describes, *relatively to the other*, an equilateral triangle whose side is to that of ABC as $\sqrt{3}:1$.

142. ABC is a triangle whose side BC is vertical. Prove that, if the times of falling from rest down the other two sides are equal, the triangle must be isosceles or right-angled.

143. If $F=ma$, and the unit of force be $\frac{1}{4}$th of the weight of the unit of mass, find the unit of time, the unit of length being one yard.

144. A box (mass 4 ozs.) containing a ball of 2 ozs., is drawn along a smooth horizontal table by means of a string which, hanging over the edge, has a mass of 3 ozs. at the other end. Find the horizontal pressure between the ball and the side of the box.

145. A cord, 50 feet, connecting two masses of 5 lbs. and 7 lbs., passes over a smooth pulley 30 feet above the horizontal plane on which the greater mass rests. Find the work which would be done in drawing the lesser body down to this plane.

146. *ABCD* is a vertical circle, and *AB* (5 feet long) is its vertical diameter. Masses of 2 lbs. and 3 lbs. fall down the chords *AB*, *AD*, whose lengths are 3 feet and 4 feet long. Compare their momenta, and also their K. E.'s at the moment when they arrive at the circumference.

147. A train is moving at the rate of 30 miles per hour when the steam is turned off. If the friction be equivalent to a retarding force $=\frac{1}{400}$th of the weight of the train, find *how long* and *how far* the train will run on the level before it stops.

148. Show that the height to which a bullet can be projected from a gun varies directly as the square of the impelling force and inversely as the square of the mass of the projectile.

149. A man walking three miles an hour walks for 40 minutes East, then for 20 minutes North, then for 30 minutes North-East, then for 60 minutes North-West. He now meets another man who, having started $1\frac{1}{2}$ hours after him from the same point, has walked in a straight line. Find the speed of the second man.

150. A train is going horizontally at 25 miles per hour. A rifle is pointed from it at a bird seen in a direction inclined at 60° to that of the train's motion, and a bullet is discharged. Supposing the bullet to move horizontally and uniformly until it kills the bird, which is flying uniformly and horizontally in a plane at right angles to the line of the train's motion; find the bird's speed, and its velocity relative to the train.

151. Express in poundals the tension of the coupling between the engine and the first carriage of a train of 100 tons which is being drawn along with an acceleration of 3 f.-s.-s.; and find the constant horizontal force which would stop the train in one minute, if the steam be shut off when a speed of 50 miles per hour has been reached.

152. A 10-ton mass falls through a height of 6 feet, and makes an impression upon a mass of iron to the depth of an inch. Find the average pressure in pounds weight exerted during the compression.

153. A quantity changes in time at a rate which is uniformly accelerated; prove that at the time t its value may be expressed by the formula

$$a+ut+\tfrac{1}{2}\alpha t^2,$$

where a is the initial value of the quantity, u its initial rate, and α the acceleration.

The values at 9^h, 12^h, 15^h being respectively 179, 299, 239, find its value at $11\frac{1}{2}^h$.

154. Express the Second Law of Motion in the form of an equation, explaining the units corresponding to each symbol employed.

155. Given the velocity (u) and angle of projection (α) of a body, find its position and velocity after a given time (t).

156. A particle is projected vertically upwards with a momentum equal to that of 1 lb. moving at the rate of 1 mile in 8 minutes, and it attains a height of 484 feet; find the mass of the particle.

157. A mass of 1 cwt., originally at rest, is lifted 100 feet by a constant force, and has at the end of the time a velocity of 20 f.-s.; find the value of the constant force in poundals, and also in pounds weight.

158. In Atwood's machine, each of the equal masses being 9 ozs., and the added mass $\frac{1}{4}$ oz., the time above the ring was 5 seconds, and the velocity below the ring was 2·2 f.-s.; find the acceleration due to gravity at that place.

159. A train, whose mass is 165 tons, is drawn by an engine of 154 H. P., the resistance to motion on the level being $11\frac{2}{3}$ lbs. weight per ton. Find the greatest speed attainable, and show that, when half the greatest speed is reached, the acceleration is $\frac{1}{8}$ f.-s.-s.

160. Two straight railway lines make an angle of 60° with each other, and two trains are running, each at the rate of 40 miles per hour, away from the point of intersection of the lines, one on the one line and one on the other. Find the direction and magnitude of their relative velocity.

161. A mass P, descending vertically, draws a mass $2P$ up a smooth inclined plane by means of a string passing over a pulley at the top of the plane. The mass $2P$ starts from rest at the foot of the plane, and when it has travelled half-way up the plane, the mass P is suddenly brought to rest. If $2P$ just reaches the top of the plane, find the inclination of the plane.

162. From two points, not necessarily in the same horizontal plane, particles are simultaneously projected with equal velocities. If the particles meet, prove that the sum of their angles of elevation must be constant.

163. An imperfectly elastic particle is projected from a point in a vertical wall so as to impinge on another smooth wall parallel to the first, and on its return strikes the first wall horizontally. The velocity of projection is such that if the particle had been projected vertically upwards it would have risen to a height above the point of projection equal to three times the distance between the walls. Find the angle of elevation if $e=\frac{1}{2}$.

164. Find the H.P. of an engine required to draw a train of 90 tons up an incline rising 1 in 100 at the rate of 20 miles an hour, the resistance due to friction being 15 pounds weight per ton.

165. Two points are moving with given velocities (v_1 and v_2) round the circumference of a circle. Find the magnitude of their relative velocities (1) when they are at the opposite ends of a diameter, (2) when they are separated by an arc of 60°.

166. Two men start from the same point with the velocity v to run in opposite directions round a circle of length $6v$. If one move uniformly, and the other increase his pace with a constant acceleration v, show that, when they meet, one will have run twice as far as the other.

167. Two perfectly elastic spheres of equal mass are travelling in parallel directions, and one overtakes the other, the straight line joining their centres, at the instant of impact, making an angle θ with their direction of motion. If α and β be the angles which their directions of motion, after impact, make with the straight line joining their centres, prove that

$$\tan\alpha\tan\beta=\tan^2\theta.$$

168. Two sides of a doubled inclined plane are each $2\frac{1}{2}$ feet long and inclined at 30° to the horizon. A string, a yard long, passes over the vertex, connecting a mass of 3 lbs. placed at the foot of one plane with a mass of 5 lbs. lying upon the other plane. Find the time in which the greater mass will slide down to the foot of its plane. If its motion be then stopped, prove that the smaller mass will just reach the vertex.

169. A particle is projected from a point P in a plane inclined at 30° to the horizon with a velocity of 6 f.-s. in a direction perpendicular to the plane, and, after passing through a point Q in the same horizontal plane as P, it strikes the inclined plane at R. Find the

lengths of the straight lines *PQ* and *PR*, and show that the time of moving from *Q* to *R* is one-third of the time in moving from *P* to *Q*.

170. Two bodies, of masses m and $2m$ respectively, are connected by a string passing over a smooth fixed pulley. Find the acceleration of their C. G.

171. Two small heavy beads, each of mass m, are connected by a rod, of length 6 feet, the mass of the rod being so small that it may be neglected. The beads can slide along a smooth fine wire which is bent into the form of a circle of radius 5 feet, and is fixed in a vertical plane. The beads are at rest in the position in which the rod is horizontal and in its highest position. If the rod be slightly displaced and fall from this position, show, by consideration of the Transformation of Energy, that the velocity of each bead at the instant when the rod is vertical is 16 f.-s.

172. A uniform ladder, whose mass is $1\frac{1}{2}$ cwt., rests with one end in contact with a vertical wall and 41 feet above the ground, and the other end on the ground and 38 feet from the wall. Find the work done in pushing the lower end 28 feet nearer the wall, neglecting friction.

173. If two small pails (of masses 2 lbs. and $1\frac{1}{2}$ lbs. respectively) be connected by a string over a smooth pulley, and if into each a half-pound of mercury be poured, find the pressures upon the bottom of each pail, the bottoms being horizontal and the sides vertical.

174. If a point and a straight line be in the same vertical plane, and if the straight line be inclined at 60° to the horizon, and the point be 12 feet from the straight line; find the time of quickest descent from the point to the line.

175. A heavy uniform board, in the form of a right-angled triangle *ABC*, is suspended from the right angle *C*. If two equal particles, starting simultaneously from *C*, slide down the sides *CA*, *CB*, show that they will reach *A*, *B* at the same instant, and also that their motion will not disturb the equilibrium of the triangle.

176. Find the work done in filling a cistern, 5 feet by 4 feet and 6 feet deep, placed at the top of a house 50 feet high, by means of a force-pump at the basement.

177. A perfectly elastic ball is projected from a point on the ground, between two parallel vertical walls, separated by a distance a, and after rebounding, first from one wall and then from the other, returns

to the point of projection. If u be the velocity and a the angle of elevation, prove that

$$u^2 \sin 2a = 2ag.$$

178. A particle is projected with a velocity u at an angle of elevation a, from a point in a horizontal plane. If e be the coefficient of elasticity between the particle and the horizontal plane, find the velocity with which the particle will start after the first impact, and the loss of K. E. during the impact.

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179. Two forces are completely represented by two straight lines AB, CD; prove that their resultant is represented in magnitude and direction by 2 HK, where H, K, are the middle points of AC, BD, respectively.

Does the resultant act along HK? Examine the case where $ABCD$ is a parallelogram.

180. A uniform ladder leans against a smooth wall; prove that the horizontal thrust of the foot on the ground bears to the weight the ratio of the distance of the foot from the wall to twice the altitude of the upper end.

How does the thrust alter as a man ascends the ladder?

181. Explain how the stability and sensibility of a balance respectively vary with the position of the C. G. of the beam. Do they vary with the weight of the scale pans?

182. Explain roughly the kind of experiment you would make in order to compare the masses of two bodies, if there were no such thing as gravity.

183. A bullet whose mass is 2 ozs. is fired into a block of wood whose mass is 20 lbs., hanging by vertical strings. The block is observed to move off with a velocity of 12 f.-s. Find the original velocity of the bullet.

184. A particle falling under gravity describes 100 feet in a particular second. How long will it take to describe the next 100 feet, if the resistance of the air be neglected?

If, owing to resistance in the latter space, it takes ·9 second, find the ratio of the resistance (assumed to be constant) to the weight of the particle.

185. What do you understand by a quantity of energy *equal to* so many 'foot-pounds'? Illustrate, by examples, that energy may vary very much in form without altering in quantity. How many foot-pounds are equal to the energy of a ton mass moving a mile a minute? Write out clearly the process by which you calculate the result.

186. A uniform bar AB, of weight W, rests in a horizontal position with the end A against a rough wall, being supported by a string connecting its middle point with a point of the wall vertically above A at a distance equal to $\frac{1}{2}AB$. Prove that if a weight nW be suspended from B, the horizontal and vertical components of the reaction of the wall will be $(2n+1)W$ and nW, respectively.

187. A particle is projected under gravity, the initial horizontal and vertical velocities being u and v respectively. Prove that the horizontal range is $2uv/g$.

Show that with a proper change in the meanings of u and v, the same formula holds for the range on an inclined plane.

188. A cricket ball thrown up vertically is caught by the thrower in 7 seconds. Draw to scale a figure showing its position at the end of every second since its start.

189. A uniform bar, 10 feet long, balances over a rail, with a boy weighing three times as much as the bar hanging on to the extreme end of it. Draw a figure showing its balancing position.

190. A picture, weighing 56 lbs., is slung over a nail in the ordinary way by a cord attached to two eyes in the top horizontal bar of its frame. If the height of the nail above this bar is half the distance between the eyes, what is the tension of the cord? Under what circumstances would the tension be equal to or greater than the whole weight of the picture?

191. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to a stop by a uniform force in the space of 18 feet. What is the tension of the rope while the stoppage is occurring? (Express it in tons' weight.)

192. Draw to scale a wheel and axle by which a man, sitting in a loop at the end of the rope wound round the axle, can haul himself up by pulling at a rope round the wheel with a force only one-fifth of his weight. What weight is sustained by the pivots?

(NOTE.—*The last 5 questions were given at the Matriculation.*)

VIII. ADMISSION TO SANDHURST.

(STATICS.)

193. If two forces in one plane be represented in magnitude, direction, and lines of action by the two sides *AB* and *DC* of the quadrilateral *ABCD*, and *E* and *F* be the middle points of *AD* and *BC*, prove that the force necessary for equilibrium will be represented in magnitude and direction by $2FE$.

194. Three uniform heavy rods *AB*, *BC*, *CA*, of lengths 5, 4, 3 feet respectively, are hinged together at their extremities to form a triangle. Prove that the whole will balance with *AB* horizontal about a fulcrum distant 1½ inch from the middle point of *AB* towards *A*.

195. A chest in the form of a rectangular parallelopiped whose weight without the lid is 200 lbs., and width from back to front 1 foot, has a lid weighing 50 lbs., and stands with its back 6 inches from a smooth wall and parallel to it.

If the lid be open and leans against the wall, find the least coefficient of friction between the chest and ground that there may be no motion.

196. Resolve a force *P* acting along the diagonal of a square into two components acting along the straight lines joining one end of that diagonal with the middle points of the opposite sides.

197. A smooth sphere is kept at rest upon a plane inclined at 30° to the horizon by means of a string, as long as the radius, attached at its two ends to points in the surfaces of the plane and sphere respectively. Find the direction and tension of the string, and the pressure upon the plane.

198. A man of 13 stones takes hold of the block of a single moveable pulley with one hand, and grasps the 'fall' of the rope with the other hand which has passed under the moveable pulley and over a 'leading block.' Find the tension of the rope when his pressure on the ground is reduced to one stone.

199. Out of a rectangular lamina, *ABCD*, there is cut an isosceles triangle *APD*, having *AD* as base. Compare the altitude of this triangle with the length of *AB*, when the remainder of the lamina balances in neutral equilibrium about *P*.

(KINETICS.)

200. If the unit of space be 2 ft., what must be the unit of time in order that one pound-weight may be the unit force, the unit of mass being 1 lb.?

201. A particle is projected with a velocity of 80 f.-s. at an angle of elevation $=\tan^{-1}(3)$. Find the greatest vertical height to which it will rise, and its horizontal range.

Also find the direction of motion when the particle is 60 ft. vertically above the ground.

202. A carriage is slipped from a train when travelling at 40 miles an hour. How far will the carriage travel before coming to rest if the resistances be $\frac{1}{27}$th of the weight?

If the carriage have a mass of $4\frac{1}{2}$ tons, find the work (in ft.-lbs.) done in bringing it to rest.

203. Masses of 1 lb. and 3 lbs. connected by a light string are placed one in each of the straight arms of a smooth tube ABC, bent at B, so as to form an angle of 120°. Find the acceleration of the masses when the tube is held in a vertical plane (1) with AB horizontal, (2) with BC horizontal; and show that in each case the tension of the string is the same.

204. Two men, A and B, are walking in two roads which meet at right angles at C, A approaching and B receding from C; prove that if they are always the same distance apart, A's velocity must be to B's velocity at each instant as CB is to CA at that instant.

205. What would be the numerical measure of g (which $=32$ f.-s.-s) if the unit of space were a yard and the unit of time the time of falling from rest down a yard?

206. If a body, starting from rest, on a smooth inclined plane pass over 40 feet in the third second, find the inclination of the plane.

207. A particle is projected from the point A on the deck of a steamer going 15 miles an hour due East, at an angle of elevation of 45°, and reaches the deck again at B, 16 feet N.E. of A. Find (1) the velocity of projection relative to the steamer, (2) the time of flight, (3) the length and direction of the range in fixed space.

208. Find the velocity of G the C. G. of two spheres of masses m and m', and moving with velocities u and u' in the same straight line, and find also the relative velocities of m and m' with respect to G.

Prove that after collision, each of these relative velocities is reversed in direction and diminished in magnitude in the ratio of e to 1 where e is the coefficient of elasticity.

209. A man of 12 stones and a sack of 10 stones are connected by a rope over a fixed smooth pulley. If the man pull himself up the rope so as to diminish the acceleration which he would have if he kept still by one-half its amount, find the acceleration of the sack in this case, and prove that the acceleration upwards of the man relative to the rope will be 3·2 f.-s.-s.

IX. ADMISSION TO WOOLWICH.

(STATICS.)

210. A particle of weight W is supported within a smooth hemispherical bowl by a string of given length having one end attached to a point in the rim. State clearly the forces which keep the particle in equilibrium, and find their magnitude if the length of the string = the radius of the bowl, and the rim is in a horizontal plane.

211. AB is a rod one foot in length; when a weight of $\frac{1}{2}$ lb. is suspended from A, the rod balances about a point 3 inches from A, and when the same weight is suspended from B, it balances about a point 5 inches from B. Find the weight of the rod and the position of its C. G.

212. Prove that the C. G. of three equal particles placed at the angles of a triangular lamina ABC coincides with G, the C. G. of the triangle.

If the particle at A be moved to A' the foot of the perpendicular from A to BC, prove that the C. G. will move to G', the foot of the perpendicular from G to BC.

213. Two forces P and Q make angles α and β with a third force R upon opposite sides of it; find the magnitude and direction of their resultant.

If $\alpha=\beta$, and $R=2\sqrt{PQ}$, show that the resultant

$$=P+2\sqrt{PQ}\cdot\cos\alpha+Q.$$

214. A uniform ladder rests at an angle of 45° with the horizon with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ, μ' be the coefficients of limiting friction between the ladder and the ground and wall respectively, show that the least horizontal force which will move the lower extremity towards the wall

$$=\frac{W}{2}\cdot\frac{1+2\mu-\mu\mu'}{1-\mu'}.$$

215. Each of a pair of sculls has $\frac{4}{5}$ths of its length outside the row-lock, and a man sculling pulls at the handle of each with the force P. Another man thrusts an oar over the stern against the bottom of the water with the force $2P$ at an angle of 60° to the horizon. Compare their effects in propelling the boat.

216. A smooth wall is inclined at 60° to the horizon; a heavy uniform rod, AB, $4\sqrt{6}$ feet long, is in equilibrium at an angle of 45° to the wall; its lower end, A, rests on the wall, and a point in it, C, rests on a smooth horizontal rail parallel to the wall. Find the distance of C from the wall.

217. The sides AB, BC, CD, DA, of a trapezium are of lengths 54, 36, 27, 45, respectively, AB being parallel to CD. Prove that its C. G. is at a distance of 16 from AB.

218. Two smooth rings, B, C, are fixed at a distance 25 inches apart, B being 9 inches, and C 16 inches above the ground. A string, $ABCD$, passes through the rings, and supports equal weights, W, W, at the extremities, A, D. Find the resultant pressures of the string upon the rings.

219. A horizontal bar, AB, 7 feet long, is supported at its extremities, and a man of 150 lbs. weight hangs from it by his hands, one being 1 foot from A, the other 3 feet from B. Find the pressures on the supports due to the weight of the man.

220. Two levers, OA, OB, of lengths 3 and 4 feet respectively, can turn in a vertical plane about a common fulcrum, O, and their middle points are connected by a string whose length is $2\frac{1}{2}$ feet. Find the least force which applied at A will keep OB horizontal with a weight of 12 lbs. suspended from B. Find also the tension of the string.

221. How many like coins, having diameters 20 times the thickness, can be piled on a table so that their centres may be in a straight line inclined at an angle of 45° to the horizon?

(KINETICS.)

222. A mass of 2 cwt. rests on a rough plane ($\mu=\frac{1}{4}$) inclined to the horizon at 30°, and a string attached to it passes over a smooth pulley at the summit and, hanging freely, supports a mass of 1 cwt. A vertical force is applied to the latter body, just sufficient to begin to raise it. Find, in ft.-lbs., the work done in thus raising it through 50 feet.

223. Two masses are connected by a string which passes over a smooth pulley. Find their ratio that they may move with the unit acceleration, a second and a foot being the units of time and length.

224. Two spheres of masses, 10 m and 11 m, are projected from the same point, with equal velocities, but in opposite directions, along a circular groove. Where will the second impact take place, e being $=\frac{3}{4}$?

225. Two particles projected with the same velocity from O, a point on an inclined plane, pass through the same point P in the plane; show that if α, β be the angles of elevation

$$\alpha+\beta=\frac{\pi}{2}+i,$$

where i = angle of inclination of the plane.

226. A man rows across a river $\frac{1}{4}$ of a mile broad in 5 minutes, always keeping his boat at right angles to the current. On reaching the opposite bank he finds that he is $\frac{1}{2}$ a mile from the starting-point. Find the velocity of the current.

227. If a force acting on a mass of 6 lbs. will cause it to move through 9 feet in 2 seconds from rest, find the measure of the force if the unit of force be the weight of 1 ounce.

228. A particle moves with uniform velocity inside a smooth horizontal circular tube one foot in diameter at the rate of seven revolutions in a minute. Compare the horizontal pressure on the tube with the weight of the particle.

229. In the case of a single moveable pulley the free end of the string passes over a fixed pulley and supports a mass, P, greater than $\frac{1}{2}$ W, where W = the mass suspended from the moveable pulley. Find the tension of the string during the motion, the three parts of the string into which it is divided by the pulleys being parallel.

230. If the unit of force were that which acting on a mass of one ton would in 1 minute generate a velocity of 1 mile a minute, how many units of force would the weight of one ton contain?

231. A mass m, falling vertically, draws a mass M along a smooth horizontal table by means of a fine string passing over a smooth pulley at the edge of the table: if the tension in the string is half the weight of m, what is the ratio of M to m?

232. Two particles are projected with a velocity of 40 f.-s. from points 88 feet apart, the one up, and the other down, a rough plane ($\mu=\frac{1}{4}$) inclined to the horizon at an angle $\tan^{-1}(\frac{7}{24})$. Find when and where they will meet, and account for the double solution.

233. A particle hangs from a fixed point in a wall by a string of length a, find the least velocity which must be given to it in order that it may make a complete revolution without the string becoming slack.

If the string come in contact with a nail in the wall, situated in the horizontal line through the point of suspension, and at a distance b from it, find the least initial velocity in order that the particle may make a complete revolution round the nail without the string becoming slack.

234. A smooth tube ACB, consisting of two straight portions, AC, CB, with a bend at its lower point C, is fixed in a vertical plane, and a particle acted on by gravity starts from rest from a point P in the arm CA, and, passing C without change of velocity, rises in the arm CB to a point Q. Show that PQ is horizontal, and that the spaces CP, CQ are proportional to the times in which they are described.

235. Two particles slide down two straight lines, in the same vertical plane, at right angles to one another, starting simultaneously from their point of intersection; prove that their distance apart, at any time, will be equal to the distance either would have descended vertically in that time.

236. A shot of 6 lbs. leaves the muzzle of a gun of 6 cwt. in a horizontal direction with a velocity of 1000 f.-s. Find the Potential Energy of the charge in ft.-lbs.

237. Two particles start simultaneously from rest, the one down an inclined plane AC, of length 25 feet, the other down a plane BC, of length 70 feet, the heights of A, B, above the horizontal plane through C being 7 and 56 feet respectively. Find which particle will arrive at C first; and when at C, how far will it be from the other particle?

238. The trail of smoke from a steamer on a course due N. is observed to extend in the direction E.S.E., while that from another, on a course due S., with the same uniform speed, is observed to be N.N.E; determine the velocity and the direction of the wind.

239. A mass of 1 lb. is suspended by a string 2 feet long in a railway carriage. Show that when the train is moving round a curve, whose radius is 1452 feet, at the rate of 30 miles an hour, the tension of the string is increased by about $\frac{1}{72}$ oz. weight, and the horizontal displacement of the 1 lb. is 1 inch nearly.

240. A locomotive engine, which can work up to 100 H.P., is attached to a train, whose mass (including the engine) is 100 tons; assuming the total resistance to be constant and equivalent to 10 lbs. weight per ton, find the greatest speed of the train in miles per hour.

When travelling at this speed the steam is shut off; find the distance and the time in which the train would be reduced to rest by the resistance alone.

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241. Weights P, Q, R are placed at the angles of a triangle ABC. If the common C.G. of the weights coincides with the orthocentre of the triangle, prove that

$$P : Q : R = \tan A : \tan B : \tan C.$$

242. A smooth uniform beam rests with its lower end against a rough vertical wall, and with a point one quarter up its length on a smooth fixed peg. If θ is the inclination of the beam to the horizon, and $\tan \epsilon$ is the coefficient of friction, prove that

$$\theta > \frac{\pi}{4} - \frac{\epsilon}{2}, \text{ and } < \frac{\pi}{4} + \frac{\epsilon}{2}.$$

243. A uniform beam rests against a rough vertical wall, and has its lower end on an equally rough horizontal plane. If the angle of friction is ϵ, prove that the inclination of the beam to the vertical when the beam is on the point of slipping is 2ϵ.

244. A table 4 feet square and 1½ inches thick is made of wood a cubic foot of which weighs 40 lbs., and is supported by four legs of the same material symmetrically placed, each of which is 2 inches square and 3 feet long. If the lower half of one leg is removed, prove that the pressures of the remaining legs on the ground are 1⅔ lbs., 45 lbs., and 45 lbs.

245. Weights are placed at the angles of a triangle, and are proportional to the opposite sides. Prove that their common C.G. is at the centre of the inscribed circle.

246. Two uniform rods each 26 feet long, weighing 60 lbs. and 30 lbs. respectively, are hinged together at their upper extremities and stand on a smooth horizontal plane, being kept in position by a string 10 feet long joining their middle points. Prove that the tension of this string is 18¾ lbs. weight, that the reaction at the hinge is nearly

$20\frac{1}{5}$ lbs. weight, and that this force is inclined to the vertical at $\tan^{-1}(2\frac{1}{2})$.

247. A ladder 50 feet long weighing 120 lbs., whose C.G. is 20 feet from its lower end, rests with one end against a smooth vertical wall and the other on a rough horizontal plane at a distance of 14 feet from the foot of the wall. Show that the friction called into play is 14 lbs. weight.

If a weight of 20 lbs. suspended from a point 40 feet up the ladder just produces slipping, prove that the coefficient of friction is $\frac{2}{15}$.

248. A uniform beam, length 16 feet, is placed on a rough peg, and rests with its lower end against a rough vertical wall. If the coefficient of friction at the wall and at the peg is $\frac{1}{2}$, and the beam is on the point of slipping down the wall when its inclination to the wall is 45°, prove that the peg is distant $\frac{7}{\sqrt{2}}$ feet from the wall.

249. A post is kept in a vertical position by three stays, each inclined to the vertical at an angle of 30°, and each having a tension of 100 lbs. weight. Prove that the total vertical pull on the post is $150\sqrt{3}$ lbs. weight.

250. Prove that a weight of $1\frac{1}{2}$ tons can be lifted by a rope, wound on a capstan 2 feet in diameter, by 10 men, each weighing 12 stone, pushing horizontally on capstan bars 4 feet from the deck, at a distance of 8 feet from the axis of the capstan, the vertical through each man's C.G. overhanging his toes a distance of one foot.

251. Prove that an engine-driver will lose control of the train on an incline steeper than one in $n \cot \phi$, if the wheels which are breaked carry only one-nth of the gross weight of the train.

252. A and B are two pegs in a vertical plane 4 feet apart, and AB makes an angle θ with the horizontal. A thin uniform rod 12 feet long passes over the peg A and has its end under the peg B. If the coefficient of friction at each peg is $\frac{1}{2}$, show that the greatest value of θ is 45°.

253. ABC is a plane triangle, and O is any point within it. Like parallel forces act at A and B proportional to the areas BOC and COA respectively. Prove that their resultant will meet AB at the point where CO produced cuts AB.

254. A heavy body weighing 20 lbs. is just kept on a rough inclined plane by a horizontal force of 2 lbs. weight and a force of 10 lbs. weight acting along the plane. If $\mu=\frac{2}{5}$, prove that the inclination of the plane to the horizon is 2 $\tan^{-1}(\frac{25}{48})$.

255. A uniform rod ACB is supported by a string fastened to C; and a weight of 6 lbs. placed at B keeps the rod horizontal. If $AC:CB=13:7$, prove that the weight of the rod is 14 lbs., and that the tension of the string is 20 lbs. weight.

256. A plank AB, length 16 feet, hinged to the floor at A, rests with the end B on the side CD of a rough inclined plane, which is free to move along the floor AC, the angle BAC being equal to $\tan^{-1}(\frac{3}{4})$, and the inclination of the plane being twice the angle BAC. The coefficient of friction for plane and floor is $\frac{1}{3}$, and the weight of the plank is equal to the weight of the plane. Show that a man whose weight is equal to that of the plank can walk up the plank a distance of 2 feet before the plane slips.

257. Four forces, whose magnitudes are $\alpha.AB$, $\beta.CB$, $\gamma.CD$, and $\delta.AD$, act in the four sides AB, CB, CD, AD respectively of a quadrilateral $ABCD$, and are in equilibrium. Prove that $\alpha.\gamma=\beta.\delta$.

258. A rod AB is supported horizontally by strings attached to its ends, the strings being inclined to the horizon at angles α and β. The C.G. of the rod is distant a from A and b from B. Prove that

$$a \tan \alpha = b \tan \beta.$$

259. A uniform beam AB weighing 40 lbs. is hinged at A, and a weight of 30 lbs. is suspended at B. A string equal to AB in length is made fast at B and its other end is made fast to C vertically over A. If AB is 12 feet and AC is 8 feet, show that the tension of the rope is 75 lbs., and the pressure at the hinge is nearly 84 lbs. weight.

Show in a figure the direction of the latter force.

260. A string 6 inches long has one end made fast to a point A and the other end carries a small ring B. Another string is made fast to a point distant 10 inches from A and in the same horizontal line with it, and then passes through the ring B and supports a weight of 11 lbs. If θ is the angle which AB makes with the horizon, prove that

$$10 \cos^2\theta - 3 \cos\theta - 5 = O.$$

261. A uniform wire, 30 inches long, is bent so as to form three sides of a regular pentagon. Show that the distance of the C.G. of the wire from the centre of the circle inscribed in the pentagon is nearly $3\frac{7}{10}$ inches.

262. A uniform ladder rests at 45° to the ground against a rough vertical wall, and on rough horizontal ground. The coefficient of friction at the wall is $\frac{1}{3}$. If a man, whose weight is half that of the ladder, can just get up to the top before slipping begins, show that the coefficient of friction at the ground is $\frac{3}{5}$.

263. A straight uniform wire ABC is bent at B so that the angle ABC is 60°, and it is then suspended from A. Show that it will rest with BC horizontal if

$$2\,BC^2 = AB^2 + 2\,AB.BC.$$

264. A mass of 10 lbs. placed on a rough plane inclined to the horizon at 30° is just on the point of slipping down the plane. Show that the least force which will draw the body up the plane is $5\sqrt{3}$ lbs. weight, and that it must make an angle of 30° with the plane.

265. O is a point within the triangle ABC. Particles are placed at A, B, C, their masses being proportional to the areas of the triangles OBC, OCA, OAB respectively. Prove that O is the C.G. of the particles.

266. A body rests on a rough plane of inclination 30° to the horizon. A force of $3\frac{1}{8}$ lbs. weight parallel to the plane will just pull the body up the plane, and a force $\frac{5}{16}$ lb. weight, also parallel to the plane, is sufficient to pull it down. Show that the mass of the body is $2\frac{13}{16}$ lbs., and that the coefficient of friction is $\frac{11}{9\sqrt{3}}$.

267. Squares are described on the sides and hypothenuse of a right-angled triangle whose sides are 3, 4, and 5 feet respectively. Show that the C.G. of these squares is distant $2\frac{3}{25}$ feet from the side 3, and $1\frac{19}{50}$ feet from the side 4.

268. A uniform ladder 30 feet long rests at 60° to the horizontal with its base on the ground, and at 20 feet from the lower end it is supported by a smooth horizontal beam. If the ladder is just on the point of slipping, prove that the coefficient of friction at the ground must be $\frac{3\sqrt{3}}{13}$.

269. A uniform string AB, a yard long, weighing 1 oz., is fastened to a fixed point at A, and the other end hangs over a smooth peg C, one foot vertically over A. Show that the tensions at A, B, and C are $\frac{1}{3}$ oz., 0, and $\frac{2}{3}$ oz. weight respectively.

270. A uniform square is placed with its plane vertical between two smooth pegs which are in the same horizontal line and c inches apart. Prove that the square will be in equilibrium when the inclination of one of its edges to the horizon is

$$\frac{1}{2}\sin^{-1}\frac{a^2-c^2}{c^2},$$

where $2a$ inches is the length of the side of the square.

271. A uniform rod 32 inches long, weighing 20 lbs., is suspended from a peg by two strings. One string, 20 inches long, is fastened to one end of the rod, and the other, 15 inches long, is made fast to a point in the rod 7 inches from the other end. Prove that the rod rests in a horizontal position, and that the tensions are equal to the weight of 12 lbs. and 16 lbs. respectively.

272. A box rests on a rough floor with its lid, whose weight is one-sixth of the weight of the whole box, open, and resting at an angle against a rough wall. If, when the box is on the point of moving, the coefficient of friction at the ground is $\frac{1}{3}$, and at the wall is $\frac{1}{4}$, prove that the angle at which the lid is resting is $\tan^{-1}(48)$.

273. The weight of a uniform door is 100 lbs., its width is 4 feet, and the distance between the hinges is 5 feet. Prove that the horizontal force straining each hinge is 40 lbs. weight.

274. Two bars in a vertical plane are each inclined at 45° to the horizontal; the ends of a light cord can slide along these bars, being attached without friction by rings each of weight P. Over the cord a weight W is slung freely. Prove that each half of the cord will rest inclined to the vertical at an angle θ given by

$$\tan\theta = \frac{W+2P}{W}$$

and that the force sustained by each bar is $\frac{W+2P}{\sqrt{2}}$.

275. Two spheres, weighing respectively 20 lbs. and 10 lbs., rest upon two smooth inclined planes, which make angles 45° and 30° with the horizontal, and press against each other. Prove that the line joining their centres is inclined to the horizontal at the angle $\tan^{-1}\left(\frac{2\sqrt{3}-1}{3}\right)$.

276. A straight uniform plank, 20 feet long, and weighing 100 lbs., rests with its middle upon a rough horizontal cylinder of 2 feet radius,

and at right angles to the axis of the cylinder. If the angle of friction is 45°, prove that the greatest weight that can be suspended from one end of the plank without upsetting it is 18·6 lbs.

277. A rectangular board, whose sides are a feet and b feet, is supported in a vertical plane on two smooth pegs in the same horizontal line, at a distance c feet apart. If the side a makes an angle θ with the vertical, prove that

$$2c \cos 2\theta = b \cos \theta - a \sin \theta.$$

278. An endless string $ABCD$ passes through two smooth fixed rings, B and D, in the same horizontal line, and carries two smooth rings which can slide freely on the string. The lower ring, A, weighs 15 lbs., and the higher, C, weighs 13 lbs. In the position of equilibrium the depth of A below BD is to the depth of C below BD as 9 : 5; prove that the tension of the string is $8\frac{1}{3}$ lbs. weight.

279. If one extremity, A, of a rod 4 inches long rests against a smooth hemispherical bowl of 6 inches radius, and the other extremity against a smooth vertical plane passing through the centre of the bowl, and if the C.G. of the rod is 1 inch from A, prove that the inclination of the rod to the vertical is $\sin^{-1}\sqrt{\frac{11}{12}}$.

280. A stone is projected with a velocity of 120 f.-s. in a direction making an angle $\tan^{-1}(\frac{4}{3})$ with the horizontal; show that

(i) After $1\frac{1}{2}$ seconds the stone has an elevation of 45° as seen from the firing point.

(ii) After $4\frac{11}{16}$ seconds the stone will be moving at right angles to the initial direction.

281. A billiard ball strikes the cushion at an angle α. If the cushion is smooth, and e is $\frac{4}{5}$, prove that the velocity after impact is $v \cos \beta$; where v is the original velocity, and $\sin \beta = \frac{3}{5} \sin \alpha$.

282. A particle projected from a point A passes through a point B (not in the same horizontal plane) t seconds later. Prove that the particle passes over the middle point of AB at a height $\frac{1}{8} gt^2$ vertically above it.

283. Prove that a bicycle, geared up to d inches, requires $336 \frac{v}{d}$ revolutions of the cranks per minute to go v miles an hour.

284. If a projectile whose mass is x lbs. is fired with a velocity u at a body whose mass is m lbs. advancing with a velocity v, the body will retain a velocity $\frac{mv-xu}{m+x}$, if the bullet is imbedded; but a velocity $v-\frac{x}{m}(u-u')$, if the bullet perforates and retains a velocity u'.

285. An express train reduced speed from 60 to 20 miles an hour in 800 feet, the distance between the distant and home signals. Show that this distance must be increased by 100 feet to make a stop in the distance, or else that the break-power must be increased by 12½ per cent.

286. Assuming that the resistance of a steamer through the water varies as the *wetted surface* and the *square of the speed*, prove that 2% increase in length, or 6% increase of displacement, with 7% increase in the horse-power, will give 1% increase in speed to a *similar* steamer.

287. A ball of mass 3 lbs., moving with a velocity of 60 f.-s., strikes another ball of mass 2 lbs. moving in the same direction with a velocity of 6 f.-s. on a smooth plane, whose inclination to the horizontal is 30°. If $e=\frac{2}{3}$, prove that the velocities of the balls at the bottom of the plane, 32 feet from the point of collision, will be 40 f.-s. and 68 f.-s.

288. A mass of 7 lbs. impinges directly on a mass of 18 lbs. at rest, and the kinetic energy of the first mass before impact is $\frac{5}{3}$ of the sum of the kinetic energies of the masses after impact. Prove that $e=\frac{2}{3}$.

289. A straight switchback railway consists of two inclined planes rounded at their junction, so that a car loses no velocity in passing from one to the other. A car starts from a point 100 feet vertically over A, and comes to rest at a point x feet vertically over B, where A and B are points on the ground 900 feet apart. If $\mu=\frac{1}{180}$ and the resistance of the air is neglected, prove that x is $94\frac{3}{8}$ feet.

290. The maximum range on a horizontal plane is 3750 yards. Show that, if the gun be placed 5000 feet above the plane and with the same angle of elevation, the horizontal range will be increased by one-third.

291. A shot of 40 lbs., moving with a velocity of 1200 f.-s., just penetrates 6 inches of armour. Prove that, if fired at a plate of 4 inches thickness, it will emerge with a velocity of $400\sqrt{3}$ f.-s.

292. A ball falls from rest 60 feet on a floor and rebounds $33\frac{3}{4}$ feet. Show that $e=\frac{3}{4}$, and that the whole distance traversed before the ball comes to rest is $214\frac{2}{7}$ feet.

293. A fly-wheel has a mass of 3 tons, which may be regarded as distributed on its rim, the diameter of the wheel being 7 feet. When revolving at 250 times per minute, prove that the energy stored up is nearly 394 foot-tons.

294. A fly-wheel is 10 feet in diameter, and has its energy reduced by 60,000 ft.-lbs. when the revolutions are reduced from 160 to 140 per minute. Prove that the mass of the wheel is $1\frac{1}{21}$ tons.

295. A man on board ship walks round a deck-cabin on a square track 200 feet in perimeter in 40 seconds. The ship is moving at the rate of $8\frac{2}{11}$ statute miles per hour. If the man starts in the direction of motion of the ship, show that in 50 minutes he will have traversed on the earth's surface a distance of 37,500 feet.

296. If the velocity of a projectile when at its greatest height is $\sqrt{\frac{2}{5}}$ of its velocity when at half its greatest height, show that the angle of elevation is 60°.

297. A train of 112 tons travels at 25 miles per hour, the friction being 16 lbs. weight per ton. Part of the train, weighing 12 tons, becomes detached. If the force exerted by the train remains the same, show that after 50 seconds the distance between the parts of the train will be 320 feet.

298. The maximum speed of a train of 200 tons is 55 miles per hour on the level, the resistances amounting to 14 lbs. weight per ton. Prove that the horse-power is $410\frac{2}{3}$, and that at 30 miles per hour the acceleration is $\frac{1}{8}$ f.-s.-s., the engine working at full power.

299. Show that the *least* velocity (u) required to hit a flag at a height of y feet and horizontal distance from the firing point of x feet is given by the equation

$$u^2=g\left(y+\sqrt{x^2+y^2}\right).$$

300. A particle starting from rest slides down a smooth plane 24 feet long and of inclination 60° to the horizontal. If $e=\frac{1}{3}$, prove that the first range on the horizontal plane at the foot will be 12 feet.

301. A 1-oz. bullet is fired from a rifle barrel, $2\frac{1}{2}$ feet long, with a muzzle velocity of 1000 f.-s. If the force on the bullet is constant, show that the rate at which work is being done on the bullet at the instant of leaving the barrel is a little over 710 horse-power.

302. A ball projected from a point on a horizontal plane strikes the plane again at a distance a from the point of projection, and rebounds again and again. If $e=\frac{1}{2}$, show that at the beginning of the fifth hop it will be at a distance $\frac{31a}{16}$ from the firing point.

303. An engine exerting 250 horse-power and weighing 150 tons has a speed of 25 miles per hour, and an acceleration of $\frac{1}{7}$ f.-s.-s. Prove that the resistance amounts to 15 lbs. weight per ton.

304. A funicular railway has a gradient of 1 in 5. Prove that if a mass of 50 lbs. be placed on the horizontal floor of one of the carriages when starting uphill with an acceleration of 2 f.-s.-s., the horizontal and vertical components of the pressure exerted are 3·065 and 50·613 lbs. weight respectively.

305. Prove that a ball of m lbs. whirling uniformly in a circle of radius a feet, n times per second, pulls outward with a force of $\frac{4\pi^2 a\, n^2 m}{g}$ lbs. weight.

306. If the ball in the last question is whirling in a horizontal circle, being suspended by a fine thread from a point vertically above the centre, show that the point must be at a height of $\frac{g}{4\pi^2 n^2}$ above the plane of the circle, in order that the motion may go on steadily.

307. A particle is projected from a point in a horizontal plane with initial velocity u. Show that if the angle of elevation is a, its height above the plane after moving a horizontal distance x is

$$x \tan a - \frac{x^2}{2R}(1+\tan^2 a),$$

where R is the maximum horizontal range.

308. In the last example, prove that the height of an obstacle at a horizontal distance $\frac{1}{2}R$ from the firing point must not exceed $\frac{3}{8}R$ if it is possible to project the particle so as to clear the obstacle.

ANSWERS.

ANSWERS.

I. P. 4.

1. 60. **2.** 22 ; 44. **3.** 2 : 5. **4.** 900.

5. 2160. **6.** $\frac{4}{9}$. **7.** $1\frac{7}{8}$ sec. **8.** $\frac{30}{77}$ sec.

9. 2 ft. **10.** $\frac{60a}{n}$ ft. **11.** $\frac{vlt_1}{l_1 t}$. **12.** $\frac{8}{9}$.

II. P. 5.

1. 360 ft. **2.** $58\frac{3}{8}$. **3.** 2 : 1. **4.** 6 : 7.

5. 1500 ft. **6.** $7\frac{1}{6}$ miles nearly. **7.** 2179 nautical miles.

8. 40 f.-s. **9.** ·63 nearly. **10.** $16^h\ 27^m$ nearly.

11. 33 f.-s. **12.** 135. **13.** 75 days.

III. P. 8.

1. (*a*) 12 ; (*b*) 15 to the E. ; (*c*) 4 to the N. ; (*d*) 9 in first direction.

2. 21. **3.** $24\frac{2}{3}$ f.-s. **4.** $4\frac{2}{5}$ f.-s.

IV. P. 11.

1. 10 f.-s. bisecting the angle. **2.** 50 f.-s. **3.** 55 f.-s.

4. 11·6 . . . f.-s. Direction with ship's motion $= \tan^{-1}\left(\frac{75}{44}\right)$.

5. $4\sqrt{37} = 24{\cdot}3$ . . . f.-s. **6.** 15·18 f.-s. **7.** 27 ; 3.

8. $\frac{11}{\sqrt{3}}$ f.-s. **9.** 60°. **10.** 134·7 . . . f.-s

11. $30 + 5\sqrt{3}$; $30 - 5\sqrt{3}$. **12.** $\sqrt{3} : 1$. **13.** 105·1 . . .

14. 30°. **15.** 144°. **16.** $\sqrt{3}v_1$; $2v_1$.

17. $\text{Cos}^{-1}\left(\frac{5-\sqrt{17}}{4}\right)$. **18.** $\text{Cos}^{-1}\left(-\frac{13}{24}\right)$.

19. $\text{Cot}^{-1}\left(\frac{59\sqrt{3}}{5}\right)$. **20.** $\left(-\frac{5}{8}\right)$. **21.** 90°.
22. $\sqrt{2}:1$. **23.** $10\sqrt{3}$; 10. **24.** 3a.
26. Distance described by either. **27.** $t\,(v_1{}^2+v_2{}^2-2v_1v_2\cos\theta)^{\frac{1}{2}}$.

V. P. 14.

2. Yes. Euc. i. 22. **3.** 0. **4.** $\sqrt{3}$, at right angles to 5.
5. $\sqrt{117}=10{\cdot}8$ . . . **6.** 90°. **7.** $\text{Cos}^{-1}\left(-\frac{7}{12}\right)$.

VI. P. 20.

1. $40\sqrt{3}$; 40. **2.** 300 f.-s. **3.** $60\sqrt{3}$; 60.
4. $7\sqrt{2}$ miles per hour in each direction. **5.** 80.
6. 8·99 . . . ; 37·39 . . . **7.** $\frac{1}{2}\,(v_1+\sqrt{3}\,v_2)$.

VII. P. 26.

1. 7·15 . . . f.-s. N.E. nearly. **2.** v, 120° with original direction.
3. $12\sqrt{2}$, S.W. **4.** (1) 25 f.-s., 120° with original direction.
(2) $25\sqrt{2}$ f.-s., 135° with original direction.
5. $\frac{\sqrt{91}}{20}$ f.-s., making an angle with original direction =

$$\tan^{-1}\left(\frac{3\sqrt{3}}{8}\right).$$

VIII. P. 28.

2. 1. **3.** 10·47 . . . f.-s. **4.** $\frac{v}{2r}$.
5. 2π. **6.** 1 : 12 : 720. **7.** $\frac{1}{27461}$ f.-s. nearly.
8. 1536 f.-s. nearly. **9.** 11 : 216.

IX. P. 33.

1. 10. **2.** 50. **3.** $10\sqrt{13}$. **4.** 30·4...
5. 23·17 . . . **6.** $\sqrt{7}$ f.-s. **8.** S.W., $5\sqrt{2}$ miles per hour.
9. N 30° E. **11.** $\tan^{-1}(\frac{11}{80})$ with vertical.
12. (1) 10 f.-s. (2) 120 ft. (3) 16 sec.

13. 4 yards; $\frac{3}{8}$ sec. **14.** $\tan^{-1}\left(\frac{a}{b}\right)$ with the vertical.

15. 44 f.-s. **16.** NW., $4\sqrt{2}$ miles per hour.

17. Wind blew from the North with a velocity equal to that of the traveller.

19. 35 miles per hour. **20.** 40 seconds. **21.** 44 f.-s.

23. 80 miles per hour. **24.** Perp. to the line *OAB*; *v*.

X. P. 43.

1. (*a.*) $10\frac{2}{3}$; (*b*) 38400; (*c*) $21\frac{9}{11}$; (*d*) $213\frac{1}{3}$; (*e*) $1\frac{1}{3}$; (*f*) $\frac{32y^2}{x}$; (*g*) $78545\frac{5}{11}$.

2. 300 : 1. **3.** $\frac{1}{960}$. **4.** 6. **5.** 3.

6. 9 : 25. **7.** $\frac{alt_1{}^2}{l_1t^2}$. **8.** $\frac{625}{32}$. **9.** 6 sec.

10. $1\frac{1}{6}$. **11.** 1000. **12.** $\frac{3}{88}$ sec. **13.** $\frac{1}{16}$ sec.

14. $10\frac{10}{11}$ sec. **15.** $22\frac{1}{2}$ miles. **16.** (1) $\frac{176}{45}$; (2) $\frac{44}{10125}$.

17. (1) $\frac{1}{12}$ sec.; (2) $\frac{1}{3}$ ft. **18.** 120 sec.

XI. P. 46.

1. 102 f.-s. **2.** 136 f.-s. **3.** 15. **4.** 1.

5. 5 sec.

XII. P. 47.

1. 625 ft. **2.** 15 f.-s. **3.** 6 secs. **4.** 10.

XIII. P. 48.

1. 10 f.-s. **2.** 6 f.-s. **3.** 7 f.-s.-s. **4.** 25 f.-s.-s. **5.** 63 ft.

XIV. P. 51.

1. 160 f.-s.; 400 ft. **2.** $976\frac{9}{16}$ ft. **3.** $32\sqrt{5}$ f.-s.

4. 256 f.-s. **5.** 3 f.-s.-s.; 2 sec.

6. $\frac{1}{4}$ sec. nearly, or $12\frac{1}{4}$ sec. nearly. **7.** 0.

8. 10 f.-s.-s. **9.** 1 sec.; 39 sec. **10.** 6 sec. **11.** 100 f.-s.

12. 132 f.-s. **13.** 66·6 f.-s. **14.** 11 sec.

15. 16 sec. nearly. **16.** $\frac{b \pm \sqrt{b^2 + 2ag}}{g}$ sec. **17.** $\frac{128\sqrt{3}}{3}$ f.-s.

18. 4 sec. **19.** 224 f.-s. **20.** 185 f.-s.
21. 1 sec. after 2nd is let fall; 96 ft. **22.** 18 f.-s.
23. (1) 84 f.-s.; (2) $72\frac{108}{169}$ ft. from ground.
24. $240\frac{1}{4}$ ft.; $7\frac{3}{4}$ sec.

XV. P. 54.

1. 30°. **2.** $4\sqrt{5}$ sec. **3.** 30°. **4.** 30°.
5. $\frac{5\sqrt{65}}{13}=3\cdot1\ldots$ secs. **6.** 9·7 ft. **7.** $80\sqrt{2}$ f.-s. **8.** 15°.
9. $185\frac{1}{3}$ f.-s. **10.** $\frac{5}{2\sqrt[4]{3}}$ sec.; $\frac{5(\sqrt{2}-1)}{2\sqrt[4]{3}}$ sec. **11.** ½ sec. nearly.
12. 120°. **13.** $3\sqrt{2l}$ f.-s. **14.** The inclination is $\sin^{-1}\left(\frac{v^2}{2lg}\right)$.

XVI. P. 56.

1. 208. **2.** 20 f.-s.-s. **3.** 24 f.-s.-s. **4.** 11 : 23.
5. 500 ft.; $2133\frac{1}{3}$ ft. **6.** 24 f.-s.-s. **7.** 30 f.-s.; 8 f.-s.-s.
8. $2\frac{1}{2}$ sec. **9.** 256 ft. **10.** 36 ft. **11.** 3136 ft.
12. 1024 ft. **13.** 8 sec. **14.** 345·6 ft.

XVII. P. 62.

3. $\cos^{-1}\left(\frac{3-2\sqrt{2}}{2}\right)$ with vertical. **4.** 45° with the vertical.
5. 60° with the vertical. **6.** 15°.
8. $\sqrt{2rg(1-\sin\theta)}$; $\sqrt{2rg(1+\sin\theta)}$, where $ADC=\theta$; $\sin\theta=\frac{1}{3}$.

XVIII. P. 65.

2. $\sqrt{2}(\sqrt{3}-1)$ sec.

XIX. P. 65.

1. 4 : 1. **2.** 4 : 1. **3.** Art. 51. **4.** 11 : 17280.
5. $1\frac{7}{8}$ ft. **9.** $\frac{121}{12}$. **10.** (1) 2000 ft.; (2) 195 ft.
13. 10 sec. **14.** (1) 160 f.-s.; (2) 400 ft.; (3) 1·46 sec., or 8·53 sec.; (4) 80 f.-s.
15. $7\frac{1}{2}$ sec. **18.** 60 yds. **19.** 72. **20.** 10 miles.
21. 4 sec. **22.** 336 ft. **24.** $\frac{88}{16}$ ft. **25.** $10\frac{1}{3}$ ft.

26. $2\frac{1}{3}$ f.-s.-s. **27.** $\frac{44}{135}$. **28.** $g\sin^2 i$. **30.** $57\frac{1}{7}$ ft.
31. 88. **32.** $3\frac{1}{8}$ sec.; 64 f.-s. **33.** 30°.
34. In ratio 4 : 3. **35.** Arts. 11 and 56. **36.** 15 : 16; $33\frac{3}{4}$ ft.
38. 1 ft. **39.** 13; angle with the vertical $=\tan^{-1}\left(\frac{5}{12}\right)$.
42. 17 f.-s. **43.** 12 f.-s.-s.; 5 f.-s. **44.** 64 ft.; 48 f.-s.
45. 160 ft. **46.** 420 ft. **47.** No.
49. $b-a;\ \frac{3a-b}{2(b-a)}$. **50.** $n\pm\sqrt{n(n-1)}$. **51.** $\frac{11}{27}$.
52. $\frac{64}{5}$ ft.; 2 sec. **53.** 22 ft.; 15 sec.; $\frac{22}{225}$ f.-s.-s. **54.** 90 sec.
55. Art. 46. **56.** (1) 256 ft.; (2) 8 sec.
57. Meas. of Vel. not affected; of Accel. doubled. **58.** 75 f.-s.
59. 5 sec. after second ball is projected; height of 560 ft.
60. $107\frac{9}{17}$ f.-s. **61.** 6400 ft.; $26\frac{2}{3}$ sec.
62. (1) 114 ft.; (2) 144 ft. **63.** (1) 8 sec.; (2) 128 ft.; (3) 20 sec.
64. (1) 864 ft.; (2) 197·6 f.-s. low wards.
65. (1) 800 ft.; (2) 1300 ft.; (3) 900 ft.
66. (1) $2\frac{3}{4}$ sec.; (2) Just over 8 sec.; (3) 473 ft.
67. (1) 68·17 f.-s.; (2) 306·7 ft. **68.** 1260 ft. nearly. **69.** 400 ft.
70. ·68 sec. **71.** 1·6 sec. after second is projected; 284 ft.
72. 70 sec. after second is projected.
73. 1184 ft. nearly. **74.** 10 sec.; 2400 ft. **75.** $66\frac{1}{2}$ f.-s. nearly.
76. $\frac{g}{8}\left(\frac{3x-1}{x-1}\right)^2$ ft. **77.** $\frac{h}{\sqrt{3}}$ ft.; (2) $2\sqrt{\frac{h}{3g}}$ sec.
78. $\frac{15u^2}{32g}$. **79.** 6·15 miles per hour.
80. 966. **81.** $\frac{12}{\sqrt{5}}$ miles; when slower has gone 2 miles.
82. $1 : \sqrt{2}$. **83.** 100 f.-s.; 500 f.-s.; 32·2 f.-s.-s. **84.** 135 ft.
85. $\sin^{-1}(\sqrt{2}-1)^2$. **86.** $\frac{l}{n^2}, \frac{3l}{n^2}, \frac{5l}{n^2}, \ldots$
87. $\sqrt{\frac{g}{2h}}\left(\frac{l^2-h^2}{l}\right)$. **88.** 2 sec. after the particle leaves *A*.
89. $\sin^{-1}(\frac{1}{2})$; $\sin^{-1}(\frac{1}{4})$. **90.** $\frac{a}{t}+\sqrt{2ag}$. **91.** $\frac{n^2}{4m}$.
92. $\frac{t(2v+gt)}{2(v_1-v-gt)}$. **93.** $v_1=v+gt$. **95.** 787 f.-s.
96. $x\mp\sqrt{x^2-2x}$; $g\sqrt{x^2-2x}$; $x=2$.
99. Dist. up Hypoth. = Base.

100. 0; $44\sqrt{3}$ f.-s. **101.** 2 : 1 **102.** $\frac{1}{2}\sqrt{8hg+n^2g^2}$.
103. $\frac{v_2-v_1}{2(t_2-t_1)}\cdot t^2$. **105.** $3AE$. **106.** $\sqrt{11}$.
108. $\frac{v\sin\theta}{t}$. **114.** $8\frac{1}{4}$ ft. **115.** 12 miles per hour.
117. $1\frac{19}{21}$ miles; $1\frac{11}{14}$ miles. **118.** $\frac{1}{2400}$ f.-s. **120.** $\tan^{-1}(\frac{3}{4})$.
122. 267 ft. nearly. **126.** (1) $\frac{1}{3600}$ f.-s.; (2) $\frac{1}{10800}$ f.-s.

XX. P. 83.

1. 150. **2.** (1) 2304; (2) $1728\sqrt{2}$. **3.** 322560.
4. 16 : 25. **5.** 86240. **6.** $17\frac{1}{64}$ ft. **7.** 4800.
8. 16000. **9.** $600\sqrt{14}$. **10.** $6\frac{2}{3}$ lbs. **11.** $4\frac{49}{144}$ ft.
12. 22400; 12800; 0; −6400. **13.** 200,000 : 1. **14.** $13333\frac{1}{3}$.
15. 5 : 24.

XXI. P. 95.

1. $12\frac{1}{2}$ lbs. weight. **2.** 72 poundals. **3.** $1\frac{3}{5}$ f.-s.-s.
4. The latter. **5.** 320 ft. **6.** 14 ft.
7. $821\frac{1}{3}$ lbs. weight. **8.** 112 lbs. weight. **9.** $2\frac{1}{2}$ sec.; 192 f.-s.-s.
10. $4\frac{4}{5}$ lbs. **11.** 293 ft. nearly. **12.** $4\frac{1}{6}$ lbs. weight.
13. 40. **14.** 5 sec. **15.** 10. **16.** $2\frac{1}{2}$ lbs. weight.
17. 900 poundals. **18.** $3\frac{3}{8}$ lbs. weight. **19.** 12 poundals.
20. 300 poundals; $9\frac{3}{8}$ lbs. weight. **21.** 11 : 1080.
22. $56\frac{1}{4}$ lbs. wt.; 40 f.-s. **23.** $13\frac{1}{2}$ lbs. weight. **24.** 253 : 48.
25. $85\frac{5}{7}$ tons. **26.** 7 : 15; 7 : 15. **27.** 33 : 25.
28. 2 : 1; 1 : 2.

XXII. P. 100.

1. 3840, *i.e.* it is doubled. **2.** $\frac{112}{3}$ poundals, or $1\frac{1}{6}$ lbs. wt.
3. 25 sec. **4.** $18\frac{3}{4}$ sec. **5.** 64 poundals, or 2 lbs. weight.
6. $68\frac{4}{7}$ sec. **7.** $14\frac{1}{10}$ sec. **8.** $1140\frac{20}{27}$ lbs. weight.
9. 25 lbs. wt. **10.** 30 minutes. **11.** 280 sec.
12. $46\frac{2}{3}$ sec. **13.** $630\sqrt{2}$ f.-s. **14.** 171,200 pndls., or 5350 lbs. wt.
15. 44,800,000 poundals, or 1,400,000 lbs. weight.
16. (1) 150 poundals, or $4\frac{11}{16}$ lbs. weight; (2) 7500 poundals, or $234\frac{3}{8}$ lbs. weight.

XXIII. P. 103.

1. 400 f.-s. **2.** 1800 ft. **3.** $3\sqrt{3} : 5\sqrt{2}$. **4.** 225 : 49.
5. 27 : 28. **6.** $\sqrt{3} : 10$. **7.** 8 : 5. **8.** 400 f.-s.
9. $12\frac{1}{2}$. **10.** Unit Momentum towards the centre of the figure.
11. $\frac{1}{8} F$. **12.** 384 units of impulse.
13. 960 units of impulse.
14. (1) 780 units of impulse;
(2) 24,960 poundals, or 790 lbs. weight.
15. 7000 m poundals, or $218\frac{3}{4}$ m lbs. weight.
16. 2,736,000 units of impulse.

XXIII. (2) P. 106.

1. $\frac{3}{4}$ sec. **2.** 56 pndls. **3.** 5 ft. **4.** $\frac{1}{80}$ pndl.
5. 2 sec. **6.** 160 lbs. **7.** $182\frac{6}{7}$ ft.; 20 sec.
8. 240 : 11. **9.** $\frac{7}{18}$. **10.** $80\frac{16}{35}$ units of space.
11. 160 units of velocity. **12.** $4\frac{1}{2}$ sec.

XXIV. P. 115.

1. $109\frac{1}{11}$ ft. **2.** $266\frac{2}{3}$ ft. **3.** 128 ft.; 30 ft.
4. $92\frac{4}{13}$ ft.; $73\frac{11}{13}$ f.-s. **5.** $21\frac{1}{3}$ ft.; $10\frac{2}{3}$ ft. **6.** $\frac{1}{4}$ sec.; 1 ft.
7. $\frac{32\sqrt{7}}{3}$ f.-s. **8.** $2\sqrt{3}$ f.-s.
9. $4\frac{12}{13}$ f.-s.-s.; $61\frac{7}{13}$ ft.; $36\frac{12}{13}$ ft.; $24\frac{8}{13}$ f.-s. **10.** $\frac{32}{13}$ f.-s.-s.
11. 3·7 sec. **12.** 27 ft. **13.** $17\frac{7}{9}$ ft.
14. $\sqrt{3}$ sec. **15.** 20 lbs. weight.
16. 32 f.-s.; 4 oz. will describe 768 ft.; the 12 oz. will describe 192 ft.
17. $10\frac{2}{3}$ f.-s.-s.; $13\frac{1}{3}$ lbs. weight. **18.** 352 ft.
19. 30 ft. from the top. **20.** 80 ft. **21.** $48\sqrt{2}$ f.-s.
22. $6\frac{1}{8}$ ft. **23.** 2·7 sec. **24.** $\frac{32}{3}$ f.-s.-s.
25. $10(\sqrt{2}+1)$ lbs. **26.** $\sin^{-1}(\frac{1}{6})$. **27.** $\sin^{-1}(\frac{1}{20})$.
28. $\frac{16}{7}(4\sqrt{3}-3\sqrt{2}=6{\cdot}13$ f.-s.-s. nearly.
29. 10·05 f.-s.-s.; 40·2 f.-s.; 181 ft. nearly.

30. 6·07 lbs. weight; 22·2 ft. nearly.
31. $12\frac{38}{39}$ lbs. **32.** 5 : 3. **33.** 3 : 1.
34. $140\frac{7}{8}$. **35.** $20\frac{10}{17}$ lbs. **36.** $14\frac{2}{7}$ lbs.
37. $6\frac{1}{2}$ sec.; 52 ft. **38.** 224 ft. **39.** 6 lbs.
40. $m = m_1$. **41.** 1 : 2. **42.** $\frac{g}{\sqrt{3}}$.
44. $(8+2\sqrt{10})$ lbs.; $(8-2\sqrt{10})$ lbs.
45. Weight of either mass. **47.** 15 lbs.
48. $12\frac{30}{113}$ oz.; $9\frac{61}{113}$ oz. **49.** 2·6 sec. **51.** $3-2\sqrt{2} : 1$.
52. $\frac{1}{2}$ sec. **55.** $4\sqrt{2l}$, or $4\sqrt{2x}$, as $x \gtrless l$.
56. $6\frac{8}{55}$ lbs.; $5\frac{52}{55}$ lbs. **59.** 30 lbs.; or $3\frac{1}{3}$ lbs. **61.** 34 ft.
62. $\frac{480}{\sqrt{21}}$. **63.** $2\frac{10}{17}$ ft. **64.** 8 f.-s.-s.
67. 32·2 f.-s.-s. **68.** 32·17 f.-s.-s. **69.** 32 f.-s.-s.

XXV. P. 122.

1. (*a*) 3200 poundals; (*b*) 4800 poundals; (*c*) 1600 poundals; (*d*) 5600 poundals; (*e*) 0.
2. (*a*) 140 lbs. weight; (*b*) 140 lbs. weight; (*c*) 154 lbs. weight; (*d*) $87\frac{1}{2}$ lbs. weight; (*e*) 0; (*f*) 280 lbs. weight; 175 lbs. weight.
3. 0. **4.** Descending with acceleration $=\frac{1}{4}g$.
5. Descending with an acceleration $=\frac{5}{18}g$.
6. (*a*) 4 lbs. weight; (*b*) 2 lbs. weight; (*c*) 12 lbs. weight; (*d*) 8 lbs. weight.
7. 4 lbs. weight. **8.** $\frac{3}{8}g$ upwards. **9.** 7200 ft.
10. Vertical; 15 lbs. weight. **11.** Vertical; 0.
12. If $P > Q$, pressure exerted by $P = \frac{2P(Q+W)}{P+Q+2W}$.
" " $Q = \frac{2Q(P+W)}{P+Q+2W}$.
13. 64·38 oz. **14.** (1) 125; (2) 75.

XXVI. P. 125.

1. $3\frac{4}{7}$ f.-s. **2.** $4\frac{12}{25}$ f.-s. **3.** $11\frac{1}{4}$ f.-s.
4. $9\frac{7}{31}$ f.-s. **5.** 27·4...ft. **6.** 3·6...ft.

7. 32·8...ft. **8.** $256\sqrt{6}$ f.-s. **9.** 15·6...f.-s.
10. Rather more than 28 tons weight. **11.** 16·1 ft.
12. $255\frac{5}{16}$ ft.

XXVII. P. 132.

2. No. See Euc. i. 22. **5.** $\text{Cos}^{-1}(-\frac{7}{8})$. **7.** $6\sqrt{2}$.

8. If each force is P, the Force$=P\sqrt{4+2\sqrt{2}}$, and its direction makes with that of the first force $\tan^{-1}(\sqrt{2}+1)$.

10. $\dfrac{P}{\sin \widehat{QR}}=\dfrac{Q}{\sin \widehat{RP}}=\dfrac{R}{\sin \widehat{PQ}}$; $\widehat{PQ}+\widehat{QR}+\widehat{RP}=360°$.

11. 2. **14.** $2\sqrt{2}$. **15.** 27 ; 7. **16.** $\sqrt{2}+1 : 1$.

19. 120°, in same line as 60. **20.** $\text{Cos}^{-1}(-\frac{7}{15})$.

21. $56\frac{2}{3}$ and $18\frac{1}{3}$ nearly. **22.** 21 nearly, makes with line of action of 47, $\tan^{-1}\left(\dfrac{5\sqrt{3}}{41}\right)$.

24. Twice the line joining the point and intersection of the diagonals.

25. $2AD$. **27.** $\sqrt{23+8\sqrt{3}}$ **28.** $Q\sqrt{3}$.

29. $\sqrt{93}$ lbs. weight. R makes with AC, $\tan^{-1}\left(\dfrac{7\sqrt{3}}{15}\right)$.

30. If E divide CD so that $CE=2ED$, then the resultant$=3BE$.

32. $2\sqrt{5}$, and inclined to CB, $\tan^{-1}(\frac{1}{2})$.

35. (1) 0; (2) $2BC$. **38.** $2AC$.

XXVIII. P. 142.

1. 12 ; 10 in. from the 7. **2.** 4 ; 12 in. from the 10.

3. 26 ; 32 ft. beyond the 78. **5.** 4 ; 12 in. beyond the 7.

8. $19\frac{1}{8}$ to the right of the point. **9.** $7\frac{85}{87}$ to the right of the point.

10. 36 ; 24. **11.** $22\frac{2}{9}$; $17\frac{7}{9}$.

12. $\dfrac{mP}{m+n}$; $\dfrac{nP}{m+n}$; $\dfrac{ma}{n}$.

XXIX. P. 148.

2. 80. **3.** 0; 108; −108. **4.** $30\sqrt{3}$.
5. 0; 160; 0; −160. **6.** 5 : 12. **7.** $3\frac{5}{9}$ from 10; $4\frac{4}{9}$ from 8.
13. 388 lbs. weight nearly. **14.** $\frac{a}{\sqrt{2}}$ from the ground. **15.** 1 : 2.

XXX. P. 151.

1. $3\frac{8}{11}$ from same end. **3.** 6.
4. 36; $\frac{1}{2}$ ft. from the middle. **6.** $4\frac{6}{7}$ ft. from the same end.
7. Where −9 acts. **8.** The middle point.
9. $\frac{W}{2}\left(\frac{W_1+W_2-2W_0}{W_0-W_2}\right)$. **10.** 3.
11. $5\frac{1}{2}$ in. **12.** $11\frac{11}{17}$ in. from 3.
13. $\frac{3l}{2W}$ from middle point. **14.** 6.
15. 1·7 ft. nearly from the fulcrum.

XXXI. P. 167.

3. 5·1 ft.; 2·9 ft.; 2·9 ft. **8.** Yes. See Art. 202. **10.** 18 in.
12. 120. **14.** $r\left(\frac{\sqrt{3}+1}{\sqrt{2}+\sqrt{3}+3}\right)$ from centre.
15. ·47a, where a is length of side. **16.** $\frac{5\sqrt{10}}{3}$ ft.
17. 7·6 ft. nearly. **18.** 1·6 ft. nearly. **19.** $4\frac{1}{6}$ ft.
20. 3·6 ft. nearly. **21.** 7·8 in. nearly.
22. 6·7 in. nearly. **23.** 1·53 ft. nearly.
25. $2\sqrt{3}$ ft. **26.** 2·62 ft. nearly. **27.** $3\frac{1}{2}$ ft.
28. $\frac{a\sqrt{19}}{6}$ from 1; $\frac{a\sqrt{13}}{6}$ from 2; $\frac{a\sqrt{7}}{6}$ from 3.
31. $\frac{a}{9}$ from E. **32.** $\frac{2}{3}$ height from the base.

33. $\frac{2\sqrt{2}a}{35}$ from centre of original square, side of which $= a$.

34. $\frac{3}{8}$ median from base.

35. $\frac{5}{18}$ median from base.

36. $\frac{n^2-3n+2\sqrt{n}}{3(n^2-n)}$ of the median from the base.

37. Altitude of $BDC = \frac{1}{2}$ alt. of BAC.

38. $\frac{7\sqrt{3}}{5}$ inches.

40. See Ex. 35.

41. $\frac{7}{15}$ median from base.

42. Alt. of $\Delta = \frac{3-\sqrt{3}}{2}a$, if $a =$ side of square.

43. Alt. of $\Delta = \frac{3-\sqrt{3}}{2}a$, if $a =$ longer side.

44. $\frac{5\sqrt{2}}{21}$ ft. from centre of square.

45. 18·04 in.

46. $\frac{12a^2+6ab+b^2}{3(4a+b)}$.

47. 1 in.

48. $\sqrt{3} : 1$.

49. $\frac{1}{64}$ hypothenuse.

51. Coincides with C.G. of the plane triangle.

52. $r \backsim r_1$.

53. 3 in.

54. 9 in.

58. $5\frac{4}{7}$ in.

59. Art. 198.

67. $\frac{1}{8}$ hypoth.

69. 5·6 in. nearly.

70. $\frac{7a}{15}$; $\frac{4a}{15}$, where $a =$ one of the equal sides.

80. 2 ft.

XXXII. P. 177.

1. $\frac{80}{\sqrt{3}}$ lbs. wt. ; $\frac{160}{\sqrt{3}}$ lbs. wt.

2. $18\frac{3}{4}$ lbs. weight ; $31\frac{1}{4}$ lbs. wt.

3. $100\sqrt{3}$ lbs. wt.

4. $60\sqrt{2}$ lbs. wt. ; $60\sqrt{2-\sqrt{2}}$ lbs. wt. ; $60\sqrt{2-\sqrt{2}}$ lbs. wt.

5. 5 lbs. wt. ; $5\sqrt{3}$ lbs. wt. ; $5\sqrt{2-\sqrt{3}}$ lbs. wt. ; $5\sqrt{2-\sqrt{3}}$ lbs. wt.

7. Pressure $= W\sqrt{\frac{1}{2}(2-\sqrt{2})}$; Tension $= \frac{1}{2}\sqrt{2}W$; W being the weight of the rod.

8. $\frac{240}{\sqrt{73}}$ lbs. wt.

9. $35\sqrt{2}$ lbs. wt.

10. $30\sqrt{3}$ lbs. wt.

11. 60°.

12. 15 lbs. wt.

13. $\frac{10}{\sqrt{3}}$ lbs. wt.

14. 13·28 ft. nearly.

15. 113 lbs. wt. ; 138 lbs. wt.

17. $\frac{3}{8}$ ton weight. **18.** $600\sqrt{3}$ lbs.
19. $\frac{1}{2}$ weight of beam. **20.** 120 lbs. wt.; 90 lbs. wt.
21. 15 lbs. wt.; 30 lbs. wt. **23.** $\frac{1}{4}$. **24.** Vertical; $\frac{3}{4}$ wt. of rod.
26. $1 : 2\cos\frac{1}{2}\alpha$; α being the angle between parts of the string; Tension is diminished.

30. $25\sqrt{2}$ lbs. wt. **33.** $\dfrac{W}{\sqrt{4W_1^2-W^2}}a$.

XXXIII. P. 188.

2. Tension $=\frac{1}{2}W\cot\theta=$ Pressure on wall; Pressure on plane $=W$.
3. $\text{Tan}^{-1}\left(\frac{1}{8}\right)$. **5.** See Art. 218.
8. If α and β be the inclinations of the strings to horizon, the Tensions are $\dfrac{W\cos\beta}{\sin(\alpha+\beta)}$ and $\dfrac{W\cos\alpha}{\sin(\alpha+\beta)}$. Inclination of rod $=\tan^{-1}\frac{1}{2}(\tan\alpha-\tan\beta)$.
9. 30 lbs. wt.; $30\sqrt{3}$ lbs. wt.
10. 10 oz. wt.; 24 oz. wt. Inclination of rod to vertical $=\cot^{-1}\left(\frac{119}{120}\right)$.
11. Tension $=\frac{5}{8}$ lb. wt.; angle between the strings $=\sin^{-1}\left(\frac{24}{25}\right)$.
12. $\frac{13}{10}W$; $\frac{7}{10}W$.
14. (1) $\dfrac{100}{\sqrt{3}}$ lbs. wt.; (2) $\dfrac{100}{\sqrt{3}}$ lbs. wt.; (3) 200 lbs. wt.
15. $30\sqrt{3}$ lbs. wt.; $30\sqrt{39}$ lbs. wt.
16. $2\cos^{-1}\left(\dfrac{P\pm\sqrt{P^2+2(2W+W_1)^2}}{2(2W+W_1)}\right)$. **17.** $\dfrac{165}{\sqrt{7}}$ lbs. wt.
18. Tens. $=22\frac{1}{2}$ lbs. wt.; Press. $=54.8$ lbs. wt. nearly. **19.** W; $W\sqrt{2}$.
21. $W(2-\sqrt{3})$; $W(\sqrt{6}-\sqrt{2})$; Inclin. to the hor. $=\tan^{-1}\left(\dfrac{2+\sqrt{3}}{2}\right)$.
22. $\text{Tan}\,\theta=\dfrac{W}{2P}$; Press. on plane $=W$; Press. on wall $=P$.
24. Press. on plane $i=\dfrac{W\sin i_1}{\sin(i+i_1)}$; Press. on plane $i_1=\dfrac{W\sin i}{\sin(i+i_1)}$.
25. $\text{Cos}\,\theta=\pm\sqrt{\dfrac{m+n}{m+2n}}$; Press. on rim $=\dfrac{mW}{m+n}\cos\theta$; Press. at the lower end $=W\tan\theta$.

XXXIV. P. 191.

2. A couple. **3.** $\dfrac{M_2a}{M_1+M_2}$. **5.** $W\sin\dfrac{180^\circ}{\pi}$; $W\cos\dfrac{180^\circ}{\pi}$.

10. $\dfrac{3}{10}W$; $\dfrac{11}{30}W$; $\dfrac{1}{3}W$. **12.** $\sqrt{\dfrac{s(s-a)}{bc}}$; $\sqrt{\dfrac{s(s-b)}{ca}}$; $\sqrt{\dfrac{s(s-c)}{ab}}$.

13. $\frac{1}{2}W\sqrt{5}$; $\sin^{-1}(\frac{4}{5})$. **14.** $\frac{1}{2}W\cot a$.

15. $\dfrac{W\sin a}{\sin(a+\theta)}$; $\dfrac{W\sin\theta}{\sin(a+\theta)}$, where $\theta=\cot^{-1}\left\{\dfrac{(m+n)\tan a+n\cot a}{m}\right\}$.

16. $W\sec\theta$; $W\tan\theta$, where $\theta=\cot^{-1}\left\{\dfrac{(m+n)\tan a}{m}\right\}$.

17. $\tan^{-1}\left(\dfrac{m\cot i-n\cot i_1}{m+n}\right)$.

18. $\tan^{-1}\left(\dfrac{m\cot a-n\cot\beta}{m+n}\right)$, where a and β are the angles made with the vertical by the strings sustaining P and Q.

19. If a be the shorter, b the longer arm, then a makes with the horizon the angle, $\tan^{-1}\left(\dfrac{b^2\cot\theta+a^2\operatorname{cosec}\theta}{b^2}\right)$. **20.** $4\frac{2}{13}$ in.

21. If $AB=2a$, the depth$=\dfrac{W_2a}{\sqrt{4W_1^2-W_2^2}}$. **23.** $\dfrac{(a+b)(a^2+b^2)}{a^2+ab+b^2}$.

24. $\tan^{-1}\left(\dfrac{b^3\cot i_1-a^3\cot i}{a^3+b^3}\right)$, if a and b be the radii; or, angle $=\tan^{-1}\left(\dfrac{W_1\cot i-W\cot i_1}{W_1+W}\right)$, if W and W_1 be the weights of the spheres. **25.** $\frac{1}{4}$ sum of the heights.

26. $\dfrac{h}{4}\left(\dfrac{a^2+2ab+3b^2}{a^2+ab+b^2}\right)$. **27.** $k\cot\frac{1}{2}C$. **28.** $k\sin C$.

33. $\frac{1}{2}W\tan a$, where $a=$angle of either wedge.

34. $\frac{1}{2}\sqrt{7}\,W$. Angle with vertical$=\cot^{-1}\left(\dfrac{5}{\sqrt{3}}\right)$.

35. $\frac{1}{3}\sqrt{6}\,W$; $\frac{1}{3}(3\sqrt{2}-\sqrt{6})\,W$. **36.** $\dfrac{a+b}{\sqrt{2ab+b^2}}W\sin\theta$.

39. Its C. G. at $\frac{1}{4}$ length of rod from thicker end vertically under centre of sphere.

40. $2a\sin\theta\sin^2\frac{1}{2}\theta=r$. **41.** $\dfrac{(4a^2+b^2)W^2-4b^2P^2}{4abW^2}$.

XXXV. P. 201.

3. Second. **4.** Third.
6. Weight of lever might act in either of the arms.
7. $6\frac{1}{2}$ ft. ; $3\frac{1}{2}$ ft. **8.** 5 ft. ; 1 ft. **9.** 12 ft. ; $7\frac{1}{2}$ ft.
10. $1\frac{13}{32}$ ft. from the middle. **11.** $P=13\frac{1}{3}$ lbs. wt. ; $W=21\frac{7}{8}$ lbs. wt.
12. 84 lbs. weight. **13.** 49 lbs. weight. **14.** 10 in.
15. $1\frac{3}{5}$ ft. from 2 lbs. **16.** $8\frac{1}{3}$ in. ; $3\frac{1}{3}$ in. **17.** 5 ft. from *F*.
18. 12 ft. **19.** $P=3$ lbs. weight ; $W=9$ lbs. weight.
20. 6 ft. **21.** $82\frac{1}{2}$ lbs. weight. **22.** 40 lbs. wt.
23. Middle point. **24.** Middle point of rod.
25. 6 ft. **26.** $41\frac{3}{4}$ lbs. wt.
27. $13\frac{1}{3}$ ft. ; 15 lbs. wt. **28.** 1 in. from middle point.
29. $\frac{11}{16}$ ft. from *F*. **30.** $22\frac{1}{2}$ in. **31.** 30°.
32. 7·84...inches from the 6. **33.** 50 ft.
34. 57·7...lbs. weight. **35.** $7\sqrt{2} : 10$.
36. 36·178 lbs. wt. perp. to rod at 6·14 ft. from 25. **37.** $\tan^{-1}(\frac{21}{5})$.
38. $\tan^{-1}(\frac{1}{2})$. **39.** 115 lbs. weight. **40.** 120°.
41. $20\frac{1}{2}$ lbs. weight. **42.** $\cos^{-1}(-\frac{1}{8})$. **43.** 120°.
44. $2 : \sqrt{3}$. **46.** $\frac{l}{2(m+1)}$; $\frac{ml}{2(m+1)}$, from *F*.
49. $\frac{2P}{m}$. **50.** 1 : 2

XXXVI. P. 210.

2. 6 oz. weight ; 3 : 2. **4.** $8\frac{1}{3}$. **5.** He loses 1 lb.
6. $\frac{1}{2}\sqrt{95}=4{\cdot}87$...lbs. weight. **9.** 4 : 3.
10. If a and b be the arms, he loses $\frac{(a \backsim b)^2}{ab}W$. **12.** $1\frac{1}{2}$ inches.
13. $\frac{1}{3}$ in. from C.G. ; 1 in. **14.** 230 lbs. wt. ; $\frac{10}{23}$ lbs. weight.
15. $3\frac{19}{30}$ oz. weight ; 3·64 oz. weight, nearly.
16. 11 : 1, if weight of pan be neglected.
17. The movable weight must be shifted nearer to the fulcrum.
22. If W_1, W_2 be the weights, $2a$ length of beam, h distance of point of suspension from the beam, θ the inclination of beam to the horizon ; then $\tan\theta=\frac{a}{h}\left(\frac{W_1-W_2}{W_1+W_2}\right)$. If the weight of the beam $=W_1$ and k the distance of the C.G. from the point of suspension ; then, $\tan\theta=\frac{(W_1-W_2)a}{h(W_1+W_2)+kW}$.
23. Stability increased, sensibility diminished.

XXXVII. P. 213.

1. 192 lbs. wt.; 216 lbs.
2. 18.
3. 50 lbs. wt.
4. 14 in.; 6 in.
5. 270 lbs. wt.
6. $7\frac{1}{2}$.
7. 36 in.
8. 3 lbs. wt.
9. $1251\frac{3}{7}$.

XXXVIII. P. 218.

1. $1 : \sqrt{3}$.
2. 400 lbs. wt.
3. 120°.
4. 60°.
5. 20 lbs. wt.; $20(\sqrt{6}-\sqrt{2}) = 20 \cdot 7$ lbs. wt. nearly; $20\sqrt{2}$ lbs. wt.
6. $100\sqrt{2}$ lbs. wt.
7. $2\cos^{-1}(\frac{119}{169})$.
8. 64 lbs. wt.
9. 53 lbs. wt.

XXXIX. P. 221.

1. 7 lbs. wt.
2. $9\frac{3}{8}$ lbs. wt.
3. 80 lbs. wt.
4. 5.
5. 6.
6. 128.
7. 1 lb. wt.
8. 275 lbs. wt.
10. Yes.
11. $1\frac{13}{16}$ lbs. wt.
12. $2\frac{13}{16}$ lbs. wt.
13. 4 lbs. wt.
14. $\frac{123}{128}$ lbs. wt.
15. 9 lbs. wt.
16. 4 lbs. wt.
17. 4 lbs. wt.
18. 3 lbs. wt.
19. 5 lbs. wt.
20. $16P = W + 85w$.
21. 16 : 15.
22. 14 lbs. wt.

XL. P. 223.

1. $\frac{1}{6}$ ton wt.
2. $\frac{3}{8}$ ton wt.
3. 15 lbs. wt.
4. 2 lbs. wt.
5. 7.
6. 10 lbs. wt.; 6.
7. 16.
8. 156 lbs. wt.
9. $3P$ hanging from lower block.

XLI. P. 226.

1. 3.
2. 6.
3. 20.
4. 20.
5. 89·6 lbs. wt.
7. 5 lbs. wt.
8. 3 lbs. wt.
9. P = twice weight of pulley.
10. $26\frac{4}{31}$ lbs. wt.
11. 57 oz. wt.
12. 832 lbs. wt.
13. $140\frac{8}{15}$ lbs. wt.

XLII. P. 227.

1. 56 lbs. wt. **2.** $\frac{9}{16}W$. **3.** (1) 1344 lbs. wt.; (2) $157\frac{1}{2}$ lbs. wt.

4. (1) $\frac{1}{2n}W$, or $\frac{W}{2n+1}$; (2) $\frac{1}{2n+1}W$, or $\frac{1}{2n+2}W$; in both cases as the 'Standing part' is fixed to the upper or lower block.

5. 40 lbs. wt. **6.** 42 lbs. wt.

7. $2\cos^{-1}\left(\frac{W+\sqrt{W^2+288}}{24}\right)$. **9.** $\frac{2P^2}{Q\pm\sqrt{Q^2+8P^2}}$. **12.** $\frac{W}{15}$

15. $3362\frac{1}{2}$ lbs. wt. **16.** $2\frac{1}{2}$ in.; $\frac{3}{4}$ in.; 1 in. respectively.

18. Heaviest at bottom; $25\frac{1}{8}$ lbs. wt.

XLIII. P. 232.

1. 20 lbs. wt.; $20\sqrt{3}$ lbs. wt. **2.** 30°; $25\sqrt{3}$ lbs. wt. **3.** 2 : 1.

4. 30°. **5.** 24. **6.** 45°; $\sqrt{2}$. **7.** 10 lbs. wt.

XLIV. P. 233.

1. $12\sqrt{3}$ lbs. wt. **2.** $\frac{100}{\sqrt{3}}$ lbs. wt. **3.** 8 lbs. wt.

4. $37\frac{1}{2}$ lbs. wt. **5.** $17\frac{1}{2}$ lbs. wt.; $62\frac{1}{2}$ lbs. wt.

6. $\tan^{-1}(\frac{24}{143})$. **7.** $\sin^{-1}(\frac{2}{5})$; $10\sqrt{21}$.

XLV. P. 235.

1. 99·6 lbs. wt. **2.** 23° 2′. **3.** 31·65 lbs. wt.

4. 25° 52′. **5.** 105·2 lbs. wt. **6.** 43° 4′.

XLVI. P. 236.

2. 121 nearly. **3.** 57° 42′; 16·9 lbs. wt. nearly. **5.** 32 ozs

XLVII. P. 236.

1. $1 : \cos i$. **2.** $\cos\theta : \sin i$. **3.** $15\sqrt{3}$ lbs. wt.
6. $W \sin i (1 - \cos i)$. **7.** $10\sqrt{10}$ lbs. wt. **8.** $2 : 1$.
9. $\cos^{-1}(\sqrt{\frac{5}{8}})$. **11.** $\sqrt{2}P$. **12.** (1) $\cos^{-1}(\frac{5}{7})$; (2) $\sin^{-1}(\frac{1}{5})$.

XLVIII. P. 241.

1. 66. **2.** $\frac{220}{7}$. **3.** 352 lbs. wt.
4. $140\frac{4}{5}$ lbs. wt. **5.** $1\frac{5}{44}$ lbs. wt. **6.** $9\frac{6}{11}$ in.
7. 45°. **8.** $2036\frac{4}{7}$ lbs. wt. **9.** $59\frac{29}{44}$ in.
10. $15(2-\sqrt{3})=4{\cdot}035$ lbs. wt. **11.** $\frac{49}{264}$ lbs. wt. **12.** $5\sqrt{3}$.
13. $424\frac{2}{7}$ lbs. wt. **14.** 45°. **15.** $105\sqrt{3}$ lbs. wt.
16. 75. **17.** $\frac{88}{21}ax$ **18.** $50\sqrt{3}\cdot\frac{a}{r}$ lbs. wt.

XLIX. P. 243.

1. 60°. **2.** 100 lbs. weight. **3.** $60(\sqrt{6}+\sqrt{2})$ lbs. wt.

L. P. 245.

2. $472\frac{1}{2}$ lbs. **3.** $\frac{3}{5}W$; $\frac{2}{5}W$. **4.** 20 lbs. **5.** 6 feet.
6. 5 ft. from him. **7.** $\frac{4}{5}$ ft. from the middle point. **8.** 10 lbs. wt.
9. $10\frac{2}{3}$ ft. **11.** 4 in.; 12 lbs. **12.** 20 lbs.
13. $\frac{m_1(m_1-m)}{m_1^2-m_1m+m^2}\cdot g$. **14.** 12 lbs. weight; 36 lbs. wt.
16. $\frac{W\sqrt{x(2r-x)}}{r-x}$. **17.** $\frac{W}{3}\operatorname{cosec}\frac{1}{2}a$. **18.** $1 : 3$.
19. $\frac{6y}{7x}$ inches. **20.** $(3\cdot 2^{n-1}-n-1)x$. **21.** $\frac{3mm_1g}{4m+m_1}$.
22. $\frac{m-1}{mn+1}\cdot g$. **23.** 40 lbs. **24.** $89\frac{13}{19}$ lbs.
25. Of $30=\frac{19}{81}g$; of $9=\frac{9}{81}g$; of $7=\frac{29}{81}g$. Tension in upper string $=20\frac{40}{81}$ lbs. wt. Tension in lower string $=10\frac{20}{81}$ lbs. wt.
26. The accels. would be $\frac{49}{191}g$; $\frac{19}{191}g$; $\frac{79}{191}g$.
The Tensions would be $22\frac{68}{191}$ lbs. wt.; and $9\frac{171}{191}$ lbs. wt.

27. $12\frac{3}{11}$ lbs. **29.** $\frac{W}{\sqrt{3}}$, W being the weight of the sphere.

30. $\frac{1}{2}W\tan\theta$. **31.** $\frac{1}{3\sqrt{2}}W$.

32. Press. on ground $= W_1 + W_2$; Press. of upper against rod $= \frac{24}{7}W_1$. Press. of lower against rod $= \frac{24}{7}W_1$; Press. between spheres $= \frac{25}{7}W_1$.

33. Between upper and bowl = 20 lbs. wt.; between lower and base = 60 lbs. wt.

Between lower and curved portion of bowl = 80 lbs. wt.; between spheres = 40 lbs. wt.

34. Accel. of $m = \frac{mm_2 - m_1m_2 + 4mm_1}{m_1m_2 + m_2m + 4mm_1}.g$;

" of $m_1 = \frac{m_1m_2 + 3\,m_2m + 4mm_1}{m_1m_2 + m_2m + 4\,mm_1}.g$;

" of $m_2 = \frac{m_1m_2 + m_2m}{m_1m_2 + m_2m + 4\,mm_1}.g$.

Tension $= \frac{2\,mm_1m_2}{m_1m_2 + m_2m + 4\,mm_1}.g$.

35. $\frac{625}{64}g$ feet. **36.** $\frac{5}{7}$ lb.

LI. P. 255.

1. 1. **2.** $\frac{1}{\sqrt{3}}$; 12 lbs. wt. **3.** Greater than $\frac{96}{\sqrt{29}}$ lbs. wt.

4. 130 lbs. wt. **5.** $\frac{2}{3}$. **6.** $>\tan^{-1}(\frac{5}{8})$.

7. 20 lbs. wt. **11.** 25 lbs. wt. **12.** $\theta = \tan^{-1}(\mu)$.

14. 25 lbs. wt.; $25\sqrt{3}$ lbs. wt. **16.** Art. 261.

17. $W\sin 2i$, making an angle i with the plane.

18. 26 lbs. wt.; the 30 lb. is supposed to be parallel to the plane.

19. ·108. **20.** 104 lbs. wt. **22.** $W\left(\frac{1}{\sqrt{x^2+1}} + \frac{1}{m}\right)$

23. At right angles to the plane downwards; or 30° with the plane.

24. Art. 266. **25.** ·048 lbs. wt.

26. (1) 50 lbs. wt.; (2) 25 lbs. wt. **27.** $P = \infty$.

28. Art. 266; $\mu = 1$. **29.** Art. 266.

31. $\mu=\frac{W\sin i-P\cos i}{W\cos i+P\sin i}$. **32.** $\frac{1}{\sqrt{3}}$.

33. $\tan^{-1}\left(\frac{8\tan\epsilon}{16-\tan^2\epsilon}\right)$. **34.** 4 feet.

35. $i=\tan^{-1}\left(\frac{\mu_1W_1+\mu_2W_2}{W_1+W_2}\right)$. **36.** Art. 267.

37. $\frac{2\mu W}{1-\mu^2}$. **38.** 8.

39. When angle of screw $=(90°-\epsilon)$; $P=\infty$.

40. 1104 lbs. wt. **41.** 357·3 lbs. wt. **42.** $\sqrt{2}-1$.

43. $7\frac{1}{17}$ ft. **44.** $\frac{19}{36}$ length of ladder.

45. $\tan\theta=\frac{y-\mu\mu_1(x-y)}{\mu x}$. **46.** $\tan\theta=\frac{1-\mu\mu_1}{2\mu_1}$; $\frac{\mu_1 W}{1+\mu\mu_1}$; $\frac{W}{1+\mu\mu_1}$.

47. $\tan\theta=\frac{x-\mu\mu_1 y}{(x+y)\mu_1}$. **48.** $\tan\theta=\frac{x\cot(i\pm\epsilon)-y\cot(i_1\mp\epsilon)}{x+y}$.

49. $\cot^{-1}\left(\frac{\sqrt{3}\pm\mu}{2}\right)$.

50. $a=\left(\frac{W}{W_1}+1\right)x\cdot\frac{\mu}{\mu^2+1}\quad(\tan\alpha+\mu)-\frac{W}{W_1}y$.

51. $\mu=\frac{1}{\sqrt{3}}$. **54.** $\cos^{-1}\left(\frac{P_1W}{P_2\sqrt{P_1^2+W^2}}\right)$.

55. Between the vertical and the normal to the plane.

58. (1) 336 lbs. wt.; (2) $42(y+8)$ lbs. wt. **61.** $\frac{7l}{8\sqrt{2}}$.

63. $\frac{m-\mu m_1}{m+m_1}g$; $\frac{(1+\mu)(mm_1)}{m+m_1}g$.

LII. P. 264.

1. $32\frac{1}{2}$ f.-s. **2.** $12\frac{1}{2}$ f.-s. in the direction of the latter.
3. $13\frac{11}{13}$ f.-s. **4.** $4\frac{8}{13}$ f.-s. in the direction of the latter.
5. $12\frac{1}{2}$ f.-s. **6.** 4 f.-s. in the direction of the former.
7. 11 : 1. **8.** $3\frac{1}{3}$ f.-s. **9.** 25 f.-s. **10.** $17\frac{6}{7}$ f.-s.
11. One loses $\frac{2}{25}v$; the other gains $\frac{3}{25}v$. **12.** 16 f.-s.

LIII. P. 267.

1. $31\frac{7}{8}$ f.-s.; $46\frac{7}{8}$ f.-s.
2. First rebounds with a vel. $24\frac{3}{8}$ f.-s.; the other with a vel. $20\frac{5}{8}$ f.-s.
3. 1 f.-s.; 16 f.-s.

4. They rebound with vels. $16\frac{11}{18}$ f.-s., and $33\frac{1}{18}$ f.-s., respectively.
5. (1) m rebounds with double its former vel. ; $3m$ remains at rest.
(2) m ,, $\frac{11}{16}$,, ; $3m$ retains $\frac{3}{16}$ of its vel.
6. $e=\frac{1}{4}$. **7.** m rebounds with a vel. $=13\frac{1}{2}u$; $3m$ retains a vel. $=2\frac{1}{2}u$.
8. 12 f.-s. **9.** 1 : 5. **10.** 1 : 6 ; $e=\frac{1}{6}$.

LIV. P. 269.

1. 20 f.-s. **2.** $e=\frac{\sqrt{3}}{2}$. **3.** $e=\frac{1}{2}$. **4.** 122 f.-s. nearly.
5. $\sqrt{\frac{5}{6}}$; $1\frac{1}{2}$ secs. **6.** e : 1. **7.** ·39 ft. **8.** 2·56 ft.
11. $h\left(\frac{1+e^2}{1-e^2}\right)$. **12.** $2\frac{1}{2}$ sec. ; $7\frac{1}{2}$ sec.
13. 100 ft. First has a vel. = 160 f.-s. downwards ; second = 140 f.-s. downwards.

LV. P. 273.

1. Art. 276. **2.** Art. 274. **4.** 30° ; 45°.
5. $\frac{1}{4}$; 1 : 4. **6.** Art. 274, Cor. ii.
7. $2\sqrt{70}$ f.-s., and makes with plane $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$.
8. M loses $\frac{5}{6}V$; N gains $\frac{5}{6}V$. **9.** $8\sqrt{15}$ f.-s. ; ·16 sec.
10. $\frac{1}{\sqrt{3}}$; $\sqrt{\frac{4-\sqrt{3}}{2}}\,.\,u$. **12.** $v=\sqrt{\frac{2gh}{3}}$.
13. The masses are in the ratio e : 1. **14.** Art. 276.
15. $\frac{1-\tan^2 a}{2}$, where a = angle of impact. **17.** 3 : 2.
20. 1st rebounds with a vel. $\frac{5}{6}v$; 2d rebounds with a vel. $\frac{5}{9}v$.
21. 125 : 1728.
22. If $u=A$'s vel. before impact, A rebounds with $\frac{5}{8}u$; B rebounds with $\frac{7}{8}u$.
23. $\frac{v^2}{8g}$; $\frac{4v^2}{3g}$. **24.** K. E. is diminished by impact.
25. Body struck goes on; and striking body goes on, stops, or rebounds as m is greater than, equal to, or less than em_1.
26. 48,000 units of impulse.
27. 19,200 units of impulse.

28. Weight of 2316 tons.
29. 2,222,080 poundals, or 69,440 lbs. weight.
30. 2240 units of impulse.
31. 484,608 poundals, or 15,144 lbs. weight
32. 1728 units of impulse.
33. 41,280 poundals, or 1290 lbs. weight.
34. 195 lbs.
35. 31,360 lbs. weight.
36. (1) 766,080 lbs. weight. (2) 6 inches
37. 39,200,000 lbs. weight.
38. 1250 foot-poundals.
39. $15,272\frac{8}{11}$ foot-poundals.

LVI. P. 283.

1. 28,800 ft. **2.** (1) 5625 ft.; (2) $37\frac{1}{2}$ sec.; (3) $22,500\sqrt{3}$ ft.
3. (1) 20,000 ft.; (2) $50\sqrt{2}$ sec.; (3) 80,000 feet.
4. $\tan^{-1}\left(\frac{7\sqrt{3}}{25}\right)$. **5.** 1003·36 f.-s. **6.** $\frac{16875\sqrt{7}}{16}$ ft.
7. $128\sqrt{3}$ f.-s. **8.** 45°.
9. $\frac{5625\sqrt{3}}{64}=152{\cdot}1$ ft. **10.** 45°. Art. 291.
11. $16\sqrt{30}=88$ f.-s. nearly. **12.** 143112·5 ft.
13. $\tan^{-1}(4)$. **14.** 15°. **15.** 80 f.-s.
16. 360 f.-s. **18.** $\tan^{-1}(\frac{4}{3})$. **19.** 75 ft.
20. $\tan\frac{1}{2}\alpha$ sec.

LVII. P. 289.

1. ·345 sec.; 5·45 sec.; 126·83 ft.
2. 152·3 ft.; 1·86 sec. **3.** 1875 ft.
4. 317 f.-s. nearly. **5.** 3921 ft. **6.** $u\cos\alpha$; horizontal.
7. $\frac{u^2\sin\alpha}{2g}\sqrt{4-3\sin^2\alpha}$.
8. $3:2:3(2-\sqrt{2})$; *i.e.*, 1000 : 667 : 586, nearly.

9. If x and y be the horizontal and vertical distances of the point,

$$u=4\sqrt{\frac{4y^2+x^2}{y}}\,;\quad \alpha=\tan^{-1}\left(\frac{2y}{x}\right).$$

10. $2+\sqrt{3}:2$. **12.** 49 f.-s. nearly; $\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$ below the horizon.

15. $\tan^{-1}(2)$, or $\tan^{-1}(8)$. **16.** 60°; 30°.

17. 124·4 f.-s.; 5·49 sec. **18.** 10·4 sec.

19. 30°. **21.** $4x$. **22.** $e=\frac{2}{25}$.

25. $\frac{V^2\sin^2\alpha}{2g}$; $\frac{V\sin\alpha}{g}$; 261·6 ft.; 4·04 sec.

26. Art. 294 (i) and (ii). **30.** $52\frac{1}{2}$°.

32. $\cos\alpha+\sin\alpha:\cos\alpha-\sin\alpha$. **35.** $x\tan\alpha-\frac{x^2g}{2u^2\cos^2\alpha}$.

36. $\alpha=\frac{1}{2}\sin^{-1}\left(\frac{Rg}{u^2}\right)$; $\frac{2u\sin\alpha}{g}$. **37.** Angle of elevation = 45°.

LVIII. P. 296.

1. 300 ft.-lbs. **2.** 50 H.P. **3.** $94\frac{28}{33}$ H.P.

4. $22\frac{2}{5}$ H.P. **5.** $63\frac{3}{44}$ H.P. **6.** $16\frac{4}{33}$ miles per hour

7. $41\frac{2}{3}$ tons. **8.** $10\frac{6}{17}$ lbs. wt. **9.** $53\frac{1}{3}$ H.P.

10. $50\frac{25}{117}$ miles per hour. **11.** 1 in 80.

LIX. P. 298.

1. 48,000 ft.-lbs. **2.** 226,800,000 ft.-lbs.

3. 4,019,400,000 ft.-lbs. **4.** 4*h.* 51*m.* 40*s.*

5. 2,128,896,000,000 μ ft.-lbs. **6.** 8400.

7. 15 miles per hour. **8.** 27,104 ft.-lbs.

9. $403\frac{1}{3}$ H.P. **10.** 3600 ft.-lbs.

11. 1,000,000 ft.-lbs. **12.** 4500 ft.-lbs.

13. 592,192 gallons nearly. **14.** $76\frac{2}{13}$ minutes.

15. The water will rise 320 feet in the pit. **16.** $158\frac{2}{5}$ gallons.

LX. P. 306.

1. 500 ft.-pdls. **2.** 540,000 ft.-pdls. **3.** 4200 ft.-lbs.

4. 1120 ft.-lbs. **5.** 8400 ft.-lbs. **6.** 1,440,000 ft.-lbs.

7. $4039\frac{68}{225}$ ft.-lbs. **8.** 168,960 ft.-pdls.; or 5280 ft.-lbs.

9. 2560 ft.-pdls., or 80 ft.-lbs.
10. 184,320 ft.-pdls., or 5760 ft.-lbs.
11. 256 ft.-lbs. **12.** 80 ft.-lbs. **13.** 640 ft.-lbs.
14. 55,000,000 ft.-lbs. **15.** 924 ft.-lbs.

LXI. P. 307.

1. $87\frac{67}{86}$. **2.** 2·88 f.-s. nearly. **5.** 4,320,000 lbs. wt.
6. 85 ft. **7.** $245\frac{5}{11}$ H.P. **8.** 9504 ft.
10. 46657·8 foot-poundals; one part drops from the rest; the velocity of the other is doubled. **14.** $\frac{7}{40}$ ft.-lb.; 1000.
16. (1) 28,000,000 foot-pounds; (2) 250,000 foot-pounds.
19. 421,390 ergs = 1 foot-poundal.
21. Velocity in the ratio 4 : 3; Mass in the ratio 9 : 4.
22. 10,560. **23.** $4517\frac{1}{3}$ ft. **24.** 3500 foot-tons.
25. 2835 foot-tons. **26.** 10,886,400 lbs. wt. **27.** 4,092,480 ft.-lbs.
28. 100 ft. **29.** 5082 ft. **30.** 480 ft.-lbs.
31. $59\frac{11}{16}$ H.P. **32.** $133\frac{3}{5}$ H.P. **33.** (1) $1850\frac{25}{103}$ ft.
(2) $4054\frac{37}{47}$ ft.
34. 3,032,400 ft.-lbs. **35.** 12,196,800 ft.-lbs.
$292\frac{23}{81}$ ft. **37.** (1) 16,570,400 ft.-lbs.
(2) 26,302 ft. nearly.
38. $\frac{1}{7}$ nearly. **39.** $250\sqrt{37} = 1520$ f.-s.
40. $\frac{m(a+9x)}{3x}g$ pdls. **41.** 764,800 lbs. wt

LXII. P. 320.

1. 39·22 in. **2.** 39·18 in. **3.** 39·03 in.
4. 13·05 ft. **5.** 9·78 in. **6.** $\frac{5}{86}$ length.
7. $\frac{1}{20}$ in. nearly. **8.** 32·075. **9.** 13·046 ft.
10. Nearly 5. **11.** 32·202. **12.** 41·808 in.
13. Too long by $\frac{1}{180}$ of its length.
14. Too short by $\frac{1}{288}$ of its length. **15.** 32·103.
16. ·0045 in. **17.** 2·446 in. **18.** 143 : 144.
20. 183·9 sec. **21.** $1\frac{2}{3}$ miles. **22.** $2\frac{1}{4}$ miles.
23. 1159 sec. **24.** 13·19 miles. **25.** $l_1 = \left(\frac{x}{x+1}\right)$
26. $\frac{720a}{}$ **27.** 32·14. **28.** $\frac{mn^2}{n_1^2}$; 99·7.
29. $1739\frac{1}{2}$ inches. **30.** $16{\cdot}36\sqrt[3]{g}$ inches. **31.** $\frac{18}{\pi^2}$ feet.

LXIII. P. 324.

1. $8\frac{5}{8}$ lbs. wt. **2.** 219·02 lbs. wt. **3.** $\frac{121}{400}$ lbs. wt.
4. $137\frac{323}{441}$ lbs. wt. **5.** $14\frac{1}{16}$ f.-s.-s. **6.** 2·28 inches.
7. $\tan\alpha = \frac{v^2}{rg}$. **8.** $\frac{420\sqrt{2}}{11} = 54$, nearly.
9. $2032\frac{4}{5}$ lbs. wt. **10.** $\frac{1}{160}$ poundal. **11.** $15\frac{3}{11}$.
12. $\frac{35}{\sqrt{\ }}$. **13.** $18\frac{117}{175}$ lbs. wt.
14. 15·02...lbs. weight; angle with vertical $= \tan^{-1}(\frac{121}{2000})$.
15. $8\sqrt{10}$ f.-s.; 60 lbs. wt.
16. (1) $28\frac{7}{16}$ lbs. wt. (2) $18\frac{7}{16}$ lbs. wt. (3) $25\frac{15}{16}$ lbs. wt.
17. $72\frac{3}{7}$ lbs. wt.; $100\frac{3}{7}$ lbs. wt. **18.** 17 times very nearly.

LXIV. P. 328.

1. $30\frac{1}{4}$ ft. **2.** 905·1 f.-s. **3.** 1·93 sec.; Art. 105 (3).
5. (1) 3 sec.; (2) 6 sec.; (3) $24g$.
6. $3906\frac{1}{4}$ ft.; $(125)^2\sqrt{3}$ ft. **7.** Art. 121.
9. 24,000. **12.** O is at the C. G. **14.** 128 ft.; 70 lbs.
15. $\frac{(m-m_1)g+T_1}{m+m_1}$ **16.** 38,400 tons.
17. $\frac{16\sqrt{5}}{\sqrt[4]{3}}$ f.-s. **18.** $\frac{1}{1382400}$ oz. wt.
19. 2 lbs. wt. **24.** 477 f.-s. nearly; $\alpha = \tan^{-1}(\frac{5}{36})$.
25. $\frac{1}{2}\frac{P^2g}{W}$. **26.** 83,200,000; 116,480,000,000.
27. $66\frac{2}{3}$ ft. **30.** $\tan^{-1}(\frac{1}{10})$. **31.** 402 tons.
32. $_{32\cdot19}$ lbs. wt.; 32·09. **35.** 787 f.-s.
36. 3850 ft. **37.** 2 seconds after particle leaves A.
39. $14\frac{7}{80}$ lbs. **40.** $1\frac{1}{2}$ feet.
42. When at $\frac{3}{16}$ length from the end. **48.** 81·43; 162·86.
50. $\frac{2W}{\sqrt{3}}$. **51.** 60° with direction of $2P$; $\sqrt{3}.P$.
52. Plank makes with horizon the angle, $\tan^{-1}(\frac{1}{2})$.

53. Inclination of plane on which P rests $= \tan^{-1}(2)$.

55. $n=2$; $\sqrt{3}.P$.

56. If θ be inclination of rod to horizon, then $\sin\frac{1}{2}\theta = \frac{\sqrt{3}-1}{2}$.

57. 4 times the radius. **60.** $\frac{2}{3}$ wt. of triangle.

61. Art. 266. **62.** $2\frac{1}{4}$ sec.; 4 ft.; $48\sqrt{5}$.

64. $2a$; $\sqrt{2}a$ parallel to DB, where a = side of square.

65. The radii to the positions of the rings are inclined at 30° to the horizon.

67. $\frac{5a}{2\sqrt{2}}$, where a is the side of the square. **69.** $1 : \sqrt{3}$; $2Q$.

73. Art. 217. **75.** They must be at right angles.

77. If θ be the known angle, and P the known force, then the required force makes an $\angle\,(90° - \theta)$ with P, and $= P\sin\theta$.

78. 2 feet.

80. Res. = replaced force in mag. but acts in the opposite direction.

82. (1) Where the perp. to the rod through the pivot cuts the fixed line; (2) when the rod is parallel to the fixed line.

86. $1 : 2\sqrt{3}$; $\sqrt{7}.P$.

88. The middle side makes with the horizon $\tan^{-1}(\frac{3}{2})$.

89. $\frac{b^3d}{a^3-b^3}$. **92.** Dists. from sides 4, 3 are 1, $\frac{8-\pi}{6-\pi}$, respectively.

93. Chord makes with vert. diam. $\cos^{-1}\left(\frac{1}{n}\right)$.

94. Makes with vertical, $\cos^{-1}\left(\frac{3-2\sqrt{2}}{2}\right)$.

95. $W.\frac{x-h\tan\theta}{h_1\tan\theta - x}$. **96.** $\frac{P^2h}{Q^2-P^2}$. **98.** 6.

99. $\frac{\sqrt{p}}{\sqrt{p}+\sqrt{q}}.l$; $\frac{\sqrt{q}}{\sqrt{p}+\sqrt{q}}.l$.

103. Distance from remaining point $= \frac{7}{12}$ diag. of a face of cube.

107. $\frac{2v\sin\alpha}{g}(u+v\cos\alpha)$. **108.** $\frac{1}{192}$; 120 sec.

109. 600. **110.** A little over $85\frac{1}{2}$ feet. **111.** 1056 : 1.

112. $49844\frac{1}{2}$ lbs. wt.

113. (1) 1,152,000 f.-s.-s.; (2) $\frac{1}{14400}$ sec.; (3) 360,000 lbs. wt.

118. $4\frac{1}{6}$ tons wt.; $3\frac{5}{6}$ tons wt. **120.** 1 lb. wt.

121. $2(P+Q)\cos\theta$. **122.** $2\sqrt{2}$

126. $2\sqrt{3}$, makes 150° with I.   **129.** $P(8+3\sqrt{3})$.

130. $\sqrt{49-6\sqrt{3}}$.   **131.** 25 lbs. wt.

135. $3\sqrt{2-\sqrt{3}}$ lbs. wt.; 3 lbs. wt.; $3\sqrt{2}$ lbs. wt.

136. $100\left(\frac{x_2g_2-x_1g_1}{x_1g_1}\right)$.

137. $W\sqrt{2(1-\sin\theta)}$; $W\sqrt{2(1-\cos\theta)}$; $W\sqrt{2}$.

138. $7\frac{1}{12}$ ft.   **140.** $\frac{71}{81}$.

142. $\text{Sin }\theta=\frac{a}{r}\sqrt{\frac{(2r)^2-(a+b)^2}{(2r)^2-4ab}}$; $\sin\theta_1=\frac{b}{r}\sqrt{\frac{(2r)^2-(a+b)^2}{(2r)^2-4ab}}$.

143. 264 ft.   **144.** 2,240,000 lbs weight.

145. 1·96 ft.   **147.** Stopped in 4th.

149. 82481 sq. feet, or a circle whose radius is 162 feet

150. $16\sqrt{3}$ ft.; 6 lbs. wt.   **151.** $2\frac{2}{11}$ H.P.

152. 1,200,000 gallons   **153.** $37\frac{1}{2}$ miles an hour.

155. 329 58 lbs. wt.   **156.** 366·21...lbs. wt.

157. 21·06 lbs. wt.   **158.** 0·274 poundals.

160. $10\sqrt{26}m$, where m is mass of the ball in lbs. His vel $=10{\cdot}02\ldots$ f.-s, and his path makes with his original path an angle $\sin^{-1}\left(\frac{1}{100{\cdot}2\ldots}\right)$.

161. 8800 feet.   **163.** $\sqrt{\frac{2h}{g}\cdot\frac{P}{P-W}}$ seconds.

ANSWERS TO THE EXAMPLES FROM RECENT EXAMINATION PAPERS

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1. 37·36 miles an hour ; $N \tan^{-1}(3\cdot6)\, E$.

3. 80·8 f.-s. **4.** $2\sqrt{2}$, bisecting angle between 5 and 6.

5. Tension $= \frac{W}{2} \cdot \frac{s}{\sqrt{l^2 - s^2}}$; Reaction at lower end $= W$; Reaction at upper end $= \frac{W}{2} \cdot \frac{s}{\sqrt{l^2 - s^2}}$.

6. (1) $\frac{1}{10}$; (2) $\frac{11}{78}$. **7.** 96 poundals, or 3 lbs. weight.

8. An angle $= 45°$. **9.** 2240 foot-lbs.

13. $181\frac{1}{2}$ feet. **14.** Yes. **15.** $\frac{1}{44\cdot8}$ feet, very nearly.

16. $\frac{2mm'g}{2m + m'}$. **21.** Tension in upper string $= 10\sqrt{3}$ lbs. weight.
,, lower ,, $= 10$,,

23. $135\frac{5}{7}$ lbs. weight ; $14\frac{2}{7}$ lbs. weight. **24.** $\frac{1}{11}g$; $5\frac{5}{11}$ lbs. weight.

27. 24 seconds. **28.** Diminishes as body falls, and has a negative value when it descends the mine.

29. $5\frac{3}{8}$ f.-s.; $7\frac{3}{8}$ f.-s. **31.** $\frac{1}{2}$. **32.** $6\frac{1}{4}$ f.-s.; $12\frac{837}{1024}$ lbs. weight.

33. 13 lbs. weight.

34. (1) Force at upper hinge $= 62\frac{1}{2}$ lbs. weight and is horizontal.
,, lower ,, $= 162\frac{1}{2}$,, and makes with vertical $\tan^{-1}(\frac{5}{24})$. (2) Forces are equal, and each $= \frac{25\sqrt{61}}{2}$ lbs. weight.

35. $5\frac{1}{8}$ lbs. weight. **37.** $\frac{3}{8}h$. **39.** QR; QP.

41. (1) mg; (2) $m(g + a)$. **42.** 160 f.-s.; $\mathrm{Sin}^{-1}(\frac{4}{5})$. **43.** 60°.

44. String must be vertical. **45.** See p. 167. **49.** $28\sqrt{2}$ f.-s.; 28 f.-s.

52. If x and y be the lengths cut off from the sides a and b, and if θ be the angle between a and b, then distance of C.G from b $= \frac{3a^2b - xy^2}{3(2ab - xy)} \cdot \sin\theta$. Similarly the distance from the side a can be found. **53.** $\frac{141}{128}$ lbs.

54. When he had gone $\frac{3}{4}$ths of the distance.

55. Three times the distance already described.
56. $\sqrt{3}$, inclined at 30° to the 4. **57.** 12 lbs. weight. **58.** 2.
59. $20\frac{1}{4}$ feet; $\frac{3}{4}$ second; $1\frac{1}{2}$ seconds. **61.** $13\frac{1}{5}$ f.-s.; 1 : 200.
62. $7\sqrt{3}$ lbs. weight. **63.** $17\frac{1}{2}$ lbs. weight.
64. 5 lbs. weight; $\frac{1}{\sqrt{3}}$. **65.** $\sqrt{996}$ f.-s. $=31\cdot56\ldots$ f.-s.
66. $\frac{1}{7}g$; $3\frac{3}{7}$ lbs. weight. **67.** $12\frac{2}{3}$ f.-s; $62740\frac{20}{27}$ foot-lbs.
68. A couple. **69.** 6 lbs. weight. **70.** $W\tan\alpha$. When he has ascended a distance $=\frac{\mu l(W+W')\tan\theta-Wx}{W'}$, where W = weight of ladder, W' = weight of man, l = length of ladder, x = distance of its C.G. from foot, θ = inclination to horizon, and μ = coefficient of friction between ladder and ground.
72. $\frac{32}{35}$ f.-s.-s.; $1\frac{1}{35}$ lbs. weight. **73.** 18 feet; $14m$ ft. lbs., if m = mass in lbs.
74. $a\sqrt{2}$, inclined at 45° to BC, where a = side of square.
75. C.G. of wire is $\frac{r \operatorname{Sin}\frac{1}{2}\theta}{\frac{1}{2}\theta}$ from centre, where r = radius of circle.
C.G. of remainder $=\frac{a\,(3\pi-4)}{3\,(2-\pi)}$ from centre of square, a = side of square. **76.** $16\sqrt{165}$ f.-s. $=205\cdot5\ldots$ f.-s. **77.** AC.
78. $T=\frac{W}{2\sin\alpha}$, where W = weight of picture and α is the given angle.
80. $\tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$; $\frac{\sqrt{3}}{4}$. W. **82.** $130\frac{10}{11}$ f.-s.
84. $2\sqrt{2}$ towards the NE. **86.** $W\sin i$ along the plane. Force is inclined to horizon at an angle $=90°-2i$).
89. P, acting along OA. **91.** $\sqrt{3}$ making 30° with side along which 5 acts, and at a point in that side produced to 3 times its length. **93.** $AE:BE=5:3$; $CE:CF=1:4$. **94.** 3 : 4.
96. $\frac{\sqrt{3}}{7}W$; $\frac{1}{2\sqrt{3}}$. **100.** 60°; $48\sqrt{3}$ lbs. weight; 96 lbs. weight.
101. 504 lbs. weight. **102.** $8\frac{1}{2}$ ozs. **104.** 16 lbs.
105. $\frac{W}{8}+\frac{w_1}{8}+\frac{w_2}{4}+\frac{w_3}{2}$. **106.** 7 ozs.; $9\frac{1}{7}$ inches from A.
107. $11\frac{2}{3}$ ozs. weight; $\tan^{-1}(\frac{4}{3})$ with vertical.
110. F inclined at 108° with the first force.
112. 140 inches. **113.** $2:\sqrt{6}:\sqrt{3}+1$.

116. (1) $T=W$; $R=W$, and R acts along AB. (2) $T_1=3W$; $R_1=\sqrt{7}W$ and R_1 makes with AD an angle $=120^\circ-\theta$, where $\theta=\cot^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

117. The two values of θ, the inclination of the rod to the wall, are given by the equation, $a\sin^2\theta\sin(\theta\pm\epsilon)=c\cos\epsilon$.

118. Distance from each of the bounding radii $=\frac{r(\sqrt{3}+3)}{8}$.

119. $\mu=\tan 15^\circ=2-\sqrt{3}$. **120.** $4P$ inclined at 15° to AB and 45° to AD. **123.** 112 lbs. weight. **124.** $(2W+w)\,c.\theta$.

125. 1 parallel to side 5. **126.** At centre of hexagon.

127. $\frac{2}{3}$-inch from the greatest angle. **129.** $\frac{1}{10}$ length of rod from top.

130. $\frac{5-\sqrt{3}}{11}$ inches. **131.** $\frac{5}{8}$ lb. weight; $\frac{1}{8}\sqrt{97}$ lbs. weight.

132. W. **134.** $T=2\frac{3}{8}W$; $R=2\frac{7}{8}W$. **135.** $\frac{\mu(W+w)\sqrt{b^2-a^2}}{w}$.

136. $W=\frac{P+Q}{2\sin i}$; $\mu=\frac{P-Q}{P+Q}.\tan i$.

137. $\frac{1}{4}\sqrt{2r}$ seconds; $\frac{2-\sqrt{2}}{2}\sqrt{r}$ seconds, where $r=$ radius of circle.

138. While starting $=\frac{p}{g}\left(\frac{gt+v}{t}\right)$ lbs. weight.

,, stopping $=\frac{p}{g}\left(\frac{gt-v}{t}\right)$ lbs. weight.

,, in uniform motion $=p$ lbs. weight.

139. $8\sqrt{5}$ f.-s. **140.** 933800 poundals.

142. Sin $2B=\sin 2C$; $\therefore B=C$, or $B=90-C$.

143. $\frac{3}{4}$ second. **144.** $\frac{2}{3}$ oz. weight. **145.** 20 foot-lbs.

146. Momenta are as 1 : 2; $K.E$'s. are as 3 : 8.

147. 550 seconds; 12100 feet. **149.** 4·97 . . . miles an hour.

150. $25\sqrt{3}$ miles an hour; 50 miles an hour, 120° with direction of train's motion.

151. 672000 poundals; $273777\frac{7}{9}$ poundals.

152. 1635200 lbs. weight. **153.** $292\frac{1}{2}$.

155. $x=u\cos a\cdot t$; $y=u\sin a\cdot t-\frac{1}{2}gt^2$, where x and y are the coordinates of the point.

156. 1 oz. **157.** 3808 poundals; 119 lbs. wt.

158. 32·12 f.-s.-s. **159.** 30 miles an hour.

160. 40 m.-h.; 120° with direction of either train. **161.** $\text{Sin}^{-1}(\frac{1}{3})$.

163. 45°. **164.** 179·52. **165.** (1) v_1+v_2; (2) $\sqrt{v_1^2+v_2^2-v_1v_2}$.

168. 1 second. **169.** $PQ=\frac{9\sqrt{3}}{16}$ ft.; $PR=1\frac{1}{2}$ ft.

170. $\frac{1}{9}g$. **172.** 1176 ft.-lbs.

173. $\frac{4}{9}$ lb. wt.; $\frac{5}{9}$ lb. wt. **174.** 1 second. **176.** 397500 ft.-lbs.

178. $u\sqrt{\cos^2\alpha+e^2\sin^2\alpha}$; $\frac{1}{2}mu^2(1-e^2)\sin^2\alpha$.

179. Only when AC is parallel to BD.

180. Increased by $\frac{W'x\cos\alpha}{h}$, where W'=man's weight, x=distance of foot from wall, h=height of upper end, α=inclination of ladder to horizon.

181. See xxxvi. 22, and its result. **183.** 1932 f.-s.

184. 0·78 . . . second; 217 : 162. **185.** 271040 ft.-lbs.

187. If V=vel. of projection, α=angle of elevation, i=inclination of plane, then if $u\equiv\frac{V\sin(\alpha-i)}{\cos i}$; $v\equiv\frac{V\cos\alpha}{\cos i}$, the range up plane $=\frac{2uv}{g}$.

189. Rail is $1\frac{1}{4}$ ft. from boy.

190. $T=28\sqrt{2}$ lbs. wt. Equal, when angle between parts of string is 120°; greater, if this angle exceed 120°.

191. $4\frac{115}{128}$ tons wt. **192.** Radius of wheel is 4 times radius of axle; Weight of man, if mass of machine be neglected.

195. $\mu=\frac{18}{5}(2-\sqrt{3})=0·965$. **196.** Each $=\frac{P\sqrt{10}}{6}$.

197. String is horizontal; Tension $=\frac{W}{\sqrt{3}}$; Pressure $=\frac{2W}{\sqrt{3}}$.

198. 56 lbs. wt. **199.** $3-\sqrt{3}:2$. **200.** $\frac{1}{4}$ second.

201. 90 ft.; 120 ft.; 60°. **202.** 1452 ft.; 542080 ft.-lbs.

203. (1) $12\sqrt{3}$ f.-s.-s.; (2) $4\sqrt{3}$ f.-s.-s. $T=12\sqrt{3}$ poundals in each case. **205.** 2. **206.** 30°.

207. $16\sqrt{2}$ f.-s.; 1 second; 35·18 ft., $E(\cos^{-1}·947)N$.

208. $\frac{m'(u-u')}{m+m'}$; $\frac{m(u'-u)}{m+m'}$. **209.** $4\frac{38}{55}$ f.-s.-s.

210. $T=\frac{W}{\sqrt{3}}$; $R=\frac{W}{\sqrt{3}}$. **211.** 1 lb. wt.; $4\frac{1}{2}$ inches from A.

213. $(P^2+Q^2+R^2+2PQ\cos(\alpha+\beta)+2QR\cos\beta+2RP\cos\alpha)^{\frac{1}{2}}$.

215. 1 : 2. **216.** 0·732 ft.

218. Pressure at $B=\frac{6W}{5}$; Pressure at $C=\frac{8W}{5}$.

219. $96\frac{3}{7}$ lbs. wt. at A; $53\frac{4}{7}$ lbs. wt. at B.

220. 16 lbs. wt.; 20 lbs. wt. **221.** 20. **222.** $1400\sqrt{3}$ ft.-lbs.

223. 33 : 31. **224.** $\frac{1}{18}$ circumference from starting-point.

226. $\frac{22\sqrt{3}}{5}$ f.-s., or $3\sqrt{3}$ miles an hour. **227.** $13\frac{1}{2}$.

228. 121 : 14400. **229.** $\frac{3PWg}{4P+W}$. **230.** $21\frac{9}{11}$.

231. $M=m$. **232.** $1\frac{1}{4}$ secs. or $9\frac{1}{6}$ secs.; 37 above, or $332\frac{4}{9}$ feet below the lower point on plane.

233. $\sqrt{5ag}$; $\sqrt{(5a-3b)g}$. **236.** $94587\frac{3}{56}$ ft.-lbs.

237. The latter; $\frac{1}{2}$ foot. **238.** S.W.; Vel. of wind = Vel. of steamer.

240 $37\frac{1}{2}$ miles an hour; $10587\frac{1}{2}$ feet; 385 seconds.

Printed by T. and A. Constable, Printers to His Majesty
at the Edinburgh University Press